Event Valence and Subjective Probability*

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Abstract

In the world of subjective probability, there is no a priori reason why probabilities — interpreted as a willingness-to-bet—should necessarily lie in the interval $[0, 1]$. We weaken the Monotonicity axiom in classical subjective expected utility (Anscombe and Aumann, 1963) to obtain a representation of preferences in terms of an affine utility function and a signed (subjective) probability measure on states. We decompose this probability measure into a non-negative probability measure ("probability") and an additive set function on states which sums to zero ("valence"). States with positive (resp. negative) valence are attractive (resp. aversive) for the decision maker. We show how our decision theory can resolve several paradoxes in decision theory, including "hedging aversion" (Morewedge et al., 2018), the conjunction effect (Tversky and Kahneman, 1982, 1983), the co-existence of insurance and betting (Friedman and Savage, 1948), and the choice of dominated strategies in strategy-proof mechanisms (Hassidim et al., 2016). We extend our theory to allow for a non-additive willingness-to-bet, which also relaxes our earlier constraints on how valence can behave.

Keywords: Signed probabilities, non-monotonicity, indifference substitution, valence, attractive and aversive states

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1 Introduction

In a world of objective probability, where probabilities are an idealization of empirical frequencies, the probability of any event must lie between 0 and 1. This property of probability was carried over to the world of subjective probability by the pioneers, such as Ramsey (1926), de Finetti (1937), and Savage (1954), without change.

In this paper, we re-examine the world of subjective probability, once one allows that there is no fundamental reason why a subjective assessment—interpreted, as usual, as a willingness-to-bet—should lie in the same 0 to 1 range. That is, we allow for the possibility both of events with subjective probability less than 0 and of events with subjective probability greater than 1. In the language of probability theory, we admit signed probabilities. Clearly, these probabilities cannot be related to frequencies, in the way subjective probabilities are sometimes thought of as aligned with objective quantities.

The concept of "hedging aversion" provides a good starting point for our theory. Consider the question of whether a decision maker (DM) might hedge in the face of uncertainty about the outcome of a game played by their favorite sports team. Under expected utility theory, we would expect the DM to accept a well-designed bet that yields a positive monetary payoff if their team loses, to offset their disappointment, against a sufficiently small negative monetary payoff in the case of a win, which partially offsets their pleasure in this case. Instead, in two lab-in-field settings, Morewedge et al. (2018) and Kossuth et al. (2020) found that sports fans preferred no bet to one that paid off in the event of a loss, even when there was no clawback in the event of a win (Morewedge et al., 2018). Donkor et al. (2023) found, in a third lab-in-field setting, that fans bet more on wins by their favorite teams as compared with other teams about whose prospects they were neutral.

Our objective in this paper is to build a simple decision theory, by relaxing a single axiom in the conventional subjective expected utility framework, that permits hedging aversion and, at the same time, allows several other kinds of behavioral effects to occur (see below). We deliberately avoid a model tuned to a specific behavioral effect and aim for generality. Naturally, there is a cost to our approach in the form of a reduced fit. For example, we expect our theory to work better for (subjectively) small monetary stakes.

Under our axiomatization, a DM's aversion to hedging is not based on a failure to process properly the likelihoods of events, but is explainable via an intrinsic "valence"—to use the term from psychology—on states of the world. Specifically, the DM attributes a valence to states, with some states viewed as attractive and other states viewed as aversive.[1] If the DM assigns a negative

[1]From `https://dictionary.apa.org/valence`: Valence [is] the subjective value of an event, object, person, or other entity in the life space of the individual. An entity that attracts the individual has positive valence whereas one that repels has negative valence.

net probability to the event that their favorite team will lose, then they will forego a hedge—even one that exhibits upside only.

Foregoing an upside-only hedge is inconsistent with subjective expected utility (SEU), as characterized axiomatically by Anscombe and Aumann (1963). Indeed, it is inconsistent with any model satisfying the well-known axiom of Monotonicity. This axiom states that if an act g yields weakly more desirable outcomes than another act f, across all possible states of the world, then g should be weakly preferred to f. In our example, let f be the bet that pays off \$0 regardless of whether the DM's favorite team wins or loses, and g the bet that payoffs off \$20 of their favorite team loses and \$0 otherwise. Then, under Monotonicity, the DM will strictly prefer g to f, or, at the least, be indifferent. Yet, in the studies quoted, fans displayed a clear preference for avoiding the hedge (bet g).

We replace Monotonicity with a weaker axiom we call Indifference Substitution, which requires only that if acts f and g yield equally desirable outcomes, across all possible states of the world, then f and g should be indifferent. (This axiom appears in Grant and Polak, 2011, where it is called Substitution.) Our main result shows that the standard axioms for subjective expected utility, with Monotonicity replaced by Indifference Substitution, characterize a representation of preferences by a standard von Neumann-Morgenstern utility on outcomes and a signed probability measure on states. Recall that a signed probability measure is an additive set function that assigns measure 1 to the overall state space, but may assign measure greater than 1 or less than 0 to some events in the space. We refer to our representation as "signed subjective expected utility" (SSEU). Paralleling standard subjective expected utility, the utility function in our representation is unique up to affine transformations and the (signed) probability measure is unique. Notice that while our representation can be seen as a particular kind of state-dependent utility (see Remark 1), it comes with the advantage that its uniqueness properties allow us to separate and uniquely identify utility and subjective probability.

The signed probability measure ν can be decomposed into an ordinary (non-negative) probability measure p and a second additive set function γ satisfying $\gamma(S) = 0$.[2] Formally: $\nu = p + \gamma$. This decomposition allows for the separation of the probabilistic beliefs about event E, represented by $p(E)$, from $\gamma(E)$, the event's psychological "valence" to the decision-maker (DM). Of course, under conventional SEU theory, a DM's likelihood of an event and their willingness-to-bet on it coincide, so that the valence is identically 0. Ramsey (1926) observed the possibility that states might hold an intrinsic desire (or the opposite) for a DM when he wrote:

> "[T]he propositions ... which are used as conditions in the options offered may be such that their truth or falsity is an object of desire to the subject. This will be found to complicate the problem, and we have to assume that there are propositions for which

[2] Note that this is not the usual Jordan decomposition of a signed measure.

this is not the case, which we shall call ethically neutral."

In Ramsey's phrasing, our set-up allows for propositions (states) that are not ethically neutral, but, instead, may carry an intrinsic positive (attractive) or negative (aversive) valence. Our valence function is the DM's measure of "ethical non-neutrality."

We employ our SSEU theory, alongside our decomposition of signed probabilities, to offer new resolutions of several well-known anomalies in decision theory. First, we return to hedging aversion (Morewedge et al., 2018; Kossuth et al., 2020; Donkor et al., 2023) and demonstrate this effect in our model. Second, we look at the conjunction fallacy (Tversky and Kahneman, 1982, 1983) from the point of view of our decomposition $\nu = p + \gamma$. We establish and interpret a sufficient condition on the ordinary probability measure p and valence γ for the conjunction-fallacy effect to arise. Our third example goes back to the classic Friedman and Savage (1948) paradox of the co-existence of insurance and betting behavior. These authors offered a resolution that depends on changing risk attitudes with changing wealth levels. By contrast, our resolution operates at a single wealth level—so that "co-existence" becomes truly the simultaneous purchase of insurance and a lottery ticket—and works if the risk-averse DM has a sufficiently positive valence for winning the lottery. Our final application of SSEU provides an explanation of empirical evidence that some individuals choose dominated strategies in strategy-proof mechanisms, such as the well-known deferred acceptance algorithm (Hassidim et al., 2016; Dreyfuss et al., 2022; Shorrer and Sóvágó, 2023). The key to our resolution is that the DM has a sufficiently negative valence for not being matched with their first choice.

Brandenburger et al. (2024a) is another application of signed probabilities—to an examination of the status of the Agreement Theorem (Aumann, 1976) of classical epistemics in a non-classical setting. That work was motivation for the development of a fully axiomatized signed decision theory as in the current paper. In their examination, Brandenburger et al. (2024a) find that conditioning becomes much more complex in a world of signed probabilities, because an event of probability 0 may contain a sub-event of strictly positive (or negative) probability. In a companion paper to the current one (Brandenburger et al., 2024b), we develop a theory of "signed conditional probability spaces," extending the classical theory of conditional probability spaces due to Rényi (1955).

Other papers in decision theory that exhibit probabilities (or charges) that may take values outside the interval $[0, 1]$ include Dekel et al. (2001, 2007), De Waegenaere and Wakker (2001), Perea (2022), and Ke and Zhao (2023). We discuss the literature in a later section.

The organization of the rest of the paper is as follows. Section 2 lays out our decision framework, defines valence, and presents our SSEU representation. Section 3 defines our four axioms—Weak Order, Independence, Archimedean Property, and Indifference Substitution—and states our representation theorem. It offers three definitions of null events, which coincide in the case of ordinary probability but are distinct in our setting. We also define formally and characterize attractive and aversive states. Section 4 contains our resolutions of hedging aversion, the conjunction-fallacy effect, the paradox of the coexistence of insurance and betting behavior, and the puzzle of choice of dominated strategies in strategy-proof mechanisms. We go on to examine the comparative statics of valence. Section 5 extends our model by weakening Independence to obtain a signed biseparable representation of preferences. This model naturally accommodates the experimental results in Schneider and Schonger (2019), where subjects jointly violate both Monotonicity and Independence. Section 6 contains discussion of related literature and covers some conceptual matters, including the possible objection that our DM would be vulnerable to a Dutch Book (de Finetti, 1937). We also make a connection to the use, going back to Wigner (1932), Dirac (1942), and Feynman (1987), of signed probabilities in quantum physical systems. Except where indicated otherwise, the proofs of the results are found in the appendices.

2 Preliminaries

2.1 Choice Setting

Consider a finite set S of *states of the world* and a set X of *consequences*. A subset $E \subseteq S$ is called an *event*. We denote by $\mathcal{F}$ the set of all functions (called *acts*) $f : S \to X$.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the *constant* act such that $x(s) = x$ for all $s \in S$. With the usual slight abuse of notation, we thus identify X with the subset of the constant acts in $\mathcal{F}$. If $f, g \in \mathcal{F}$, and an event $E \subseteq S$, we denote by $fEg \in \mathcal{F}$ the act that yields $f(s)$ if $s \in E$ and $g(s)$ if $s \notin E$. Given $f \in \mathcal{F}$ and $u : X \to \mathbb{R}$, $u(f)$ denotes the function $s \mapsto u(f(s))$.

We assume additionally that X is a convex subset of a vector space. We can then define for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act $\alpha f + (1 - \alpha)g \in \mathcal{F}$, which yields $\alpha f(s) + (1 - \alpha)g(s) \in X$ for every $s \in S$.

2.2 Preferences

We model the decision maker's (DM's) preferences on $\mathcal{F}$ by a binary relation $\succcurlyeq$. As usual, $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succcurlyeq$. If $f \in \mathcal{F}$, an element $x_f \in X$ is a certainty equivalent for f if $f \sim x_f$.

A signed probability measure is a function $\nu : S \to \mathbb{R}$ such that $\sum_{s \in S} \nu(s) = 1$. Given $E \subseteq S$, set $\nu(E) = \sum_{s \in E} \nu(s)$. The set $\Delta(S)$ denotes the set of all signed probability measures on S, that is

$$\Delta(S) = \left\{ \nu : S \to \mathbb{R} : \sum_{s \in S} \nu(s) = 1 \right\}.$$

We denote by $\Delta^+(S)$ the set of ordinary (non-negative) probability measures on S, i.e.

$$\Delta^+(S) = \left\{ \nu : S \to [0, \infty) : \sum_{s \in S} \nu(s) = 1 \right\}.$$

Given $\phi : S \to \mathbb{R}$ and $\nu \in \Delta(S)$, set

$$\int_S \phi d\nu := \sum_{s \in S} \phi(s)\nu(s).$$

We conclude with the natural extension of the SEU model to signed probability measures.

Definition 1. We say that (u, ν) is a **signed subjective expected utility** (**SSEU**) representation of $\succcurlyeq$ if there are an affine function $u : X \to \mathbb{R}$ and $\nu \in \Delta(S)$ such that

$$f \succcurlyeq g \iff \int_S u(f) d\nu \geq \int_S u(g) d\nu.$$

2.3 The Model

We start by introducing a set function, called the *valence*, that quantifies the attractiveness or aversiveness of an event.

Definition 2. A function $\gamma : 2^S \to \mathbb{R}$ is called a **valence** if

1. For all $E, F \subseteq S$, if $E \cap F = \emptyset$, then $\gamma(E \cup F) = \gamma(E) + \gamma(F)$;

2. $\gamma(E) + \gamma(E^c) = 0$.

Note that Conditions 1 and 2 imply $\gamma(S) = 0$, $\gamma(\emptyset) = 0$, and $\gamma(A) = \sum_{s \in A} \gamma(s)$. Condition 1 says that the valence of an event depends on the valence of the states comprising the event (we relax this property in Section 5). Condition 2 says that if an event has a positive valence, its complement has a negative valence (again, we relax this assumption in Section 5). Moreover, the valence of the entire state space is 0.

Definition 3. We say that (u, p, γ) is a **valence representation of** $\succcurlyeq$ if there are an affine function $u : X \to \mathbb{R}$, $p \in \Delta^+(S)$, and a valence function $\gamma : 2^S \to \mathbb{R}$ such that

$$f \succcurlyeq g \iff \int_S u(f) dp + \int_S u(f) d\gamma \geq \int_S u(g) dp + \int_S u(g) d\gamma.$$

If one defines the set function $\nu = p + \gamma$, the properties of the probability p and of the valence γ ensure that ν is a signed measure with $\nu(S) = 1$. Indeed, both p and γ are additive over disjoint unions and $\nu(S) = p(S) + \gamma(S) = 1$, so that $\nu \in \Delta(S)$. Therefore, any valence representation can be rewritten as an SSEU representation, as defined above.

To help in further interpreting a valence representation, consider a bet xEy with $x \succcurlyeq y$. Given a valence representation (u, p, γ) of ν, its SSEU value is

$$V(xEy) = \underbrace{p(E)u(x) + (1 - p(E))u(y)}_{\text{SEU component}} + \underbrace{\gamma(E)(u(x) - u(y))}_{\text{valence component}}.$$

The value of betting on E is the sum of a classical SEU component and a non-classical valence component. Since $x \succcurlyeq y$, $u(x) \geq u(y)$, so that a positive (resp. negative) valence $\gamma(E)$ will increase (resp. decrease) the overall value of the bet xEy. In particular, if $u(x) = 1$ and $u(y) = 0$, then $V(xEy) = p(E) + \gamma(E)$, highlighting the decomposition of the DM's willingness-to-bet on E into likelihood and valence components.

It is important to observe that an event which commands a willingness-to-bet larger than 1 is not necessarily perceived by the DM as a "sure event," i.e., as an event with likelihood equal to 1. Clearly, the DM can assess $p(E) + \gamma(E) > 1$ while $p(E) < 1$. An analogous remark applies to events that command a negative willingness-to-bet.

We now show that any SSEU representation admits a valence representation.

Proposition 1. *The preferences $\succcurlyeq$ have an SSEU representation if and only if they admit a valence representation.*

We sketch the proof and leave the details to Appendix A. We have already shown sufficiency. For necessity, observe that if we define $\nu^+, \nu^- \in \mathbb{R}^S$ by

$$\nu^+(s) = \max\{0, \nu(s)\} \text{ and } \nu^-(s) = -\max\{0, -\nu(s)\},$$

for every $s \in S$, we obtain the Jordan decomposition

$$\nu(s) = \nu^+(s) - \nu^-(s),$$

for every $s \in S$. This decomposition can in turn be written as (see Proposition 4 in Appendix A)

$$\nu^+(s) - \nu^-(s) = (1+b)p^+(s) - bp^-(s),$$

for every $s \in S$, where p^+, $p^- \in \Delta^+(S)$ and $b \geq 0$. A valence representation follows once we let

$$p^* = p^+ \text{ and } \gamma^* = b(p^+ - p^-). \tag{1}$$

We call the decomposition (p^*, γ^*) thus obtained the **minimal decomposition** of ν, and the function γ^*, extended to all subsets of S using the properties of a valence, the **minimal valence** of ν. These terms reflect the fact that multiple decompositions of the signed measure ν are possible, so that there are infinitely many valence representations. As we presently show, the decomposition (p^*, γ^*) is the unique one that: (i) minimizes the overall "impact" of γ, thus providing the most classical description of the DM's choices, and (ii) at the same time preserves the likelihood ratios for states with positive willingness-to-bet.

To formalize these ideas, we introduce two additional pieces of notation. For $\nu \in \Delta(S)$, we let $P = \{s \in S : \nu(s) > 0\}$. Also, for any $\phi : S \to \mathbb{R}$, let $\|\phi\|_v$ denote the total variation of ϕ, i.e., $\|\phi\|_v = \sum_{s \in S} |\phi(s)|$.[3]

[3] We owe the proof of the first statement in Theorem 1 to Luigi Montrucchio.

Theorem 1. *Let $\nu \in \Delta(S)$ be a signed probability measure and (p^*, γ^*) be its minimal decomposition. Then*

$$p^* \in \underset{q \in \Delta^+(S)}{\operatorname{argmin}} \|\nu - q\|_v . \tag{2}$$

Moreover, within the family of decompositions (p, γ) where p satisfies (2), the decomposition (p^*, γ^*) is unique in satisfying $\nu(s)/\nu(s') = p^*(s)/p^*(s')$ for all $s, s' \in P$.

Observe that, as a consequence, we find that the minimal decomposition (p^*, γ^*) satisfies

$$\sup_{A \subseteq S} |\gamma^*(A)| \leq \sup_{A \subseteq S} |\gamma(A)|,$$

for any alternative decomposition (p, γ). Therefore, the minimal decomposition satisfies property (i) above. As for property (ii), suppose the DM observes an event E containing only states with positive willingness-to-bet; i.e., $E \subseteq P$. Then, using the definition of conditional probability, we obtain

$$\frac{\nu(s|E)}{\nu(s'|E)} = \frac{\nu(s)}{\nu(s')} = \frac{p^*(s)}{p^*(s')} = \frac{p^*(s|E)}{p^*(s'|E)},$$

for every $s, s' \in E$. By contrast, for any other decomposition (p, γ), we will have $\nu(s|E)/\nu(s'|E) \neq p(s|E)/p(s'|E)$. Thus, the likelihood p^* chosen via our minimal decomposition is the unique one satisfying (2) that is consistent with Bayesian updating of ν. The importance of this updating property of the minimal decomposition is explored further in our companion paper (Brandenburger et al., 2024b).

Next, we provide a few examples of SSEU preferences. In the first two, we employ minimal decompositions. The last two employ decompositions which are intuitive but not minimal.

Example 1. Suppose there is only one aversive state s^*, so that $\gamma^*(s) = b(p^+(s) - \delta_{s^*})$. Then

$$V(f) = \int_S u(f)dp^+(s) + b\left(\int_S u(f)dp^+(s) - u(f(s^*))\right).$$

Consider the bet xEy for some event E with $s^* \in E^c$. Then

$$V(xEy) = p^+(E)u(x) + (1 - p^+(E))u(y) + bp^+(E)(u(x) - u(y)),$$

and the willingness-to-bet on E is $\nu(E) = (1 + b)p^+(E)$. For general acts, if $f \succcurlyeq g$, then

$$\int_S u(f)dp^+ - \int_S u(g)dp^+ \geq \frac{b}{1+b}(u(f(s^*)) - u(g(s^*)).$$

Suppose $u(f(s^*)) - u(g(s^*)) \geq 0$. Then a preference for f over g implies that the SEU component of the value of f is sufficiently larger than that of g to override the negative value of obtaining more utility in the aversive state. If $u(f(s^*)) - u(g(s^*)) \leq 0$, a preference for f over g can obtain even if the SEU component of the value of g is strictly larger than that of f. △

Example 2. Paralleling Example 1, suppose that there is only one attractive state s^*, so that $\gamma^*(s) = b(\delta_{s^*} - p^-(s))$. Then

$$V(f) = u(f(s^*)) + b\left(u(f(s^*)) - \int_S u(f(s))dp^-\right).$$

Consider the bet xEy for some E with $s^* \in E$. Then

$$V(xEy) = u(x) + b(1 - p^-(E))(u(x) - u(y)),$$

and the willingness-to-bet on E is $\nu(E) = 1 + b(1 - p^-(E))$. For general acts, if $f \succcurlyeq g$, then

$$u(f(s^*)) - u(g(s^*)) \geq \frac{b}{1+b}\left(\int_S u(f)dp^- - \int_S u(g)dp^-\right).$$

△

Example 3. Consider a SSEU preference with a (non-minimal) valence given by

$$\gamma(s) = k(p(s) - 1/|S|),$$

for some $k \geq 0$ and $p \in \Delta^+(S)$. Then

$$V(f) = \int_S u(f)dp(s) + k\left(\sum_{s\in S}(p(s)u(f(s)) - \frac{1}{|S|}\right).$$

There is positive (resp. negative) valence if a state is more (resp. less) likely than the uniform case. The willingness-to-bet on an event E is $\nu(E) = p(E) + k\,(p(E) - |E|/|S|)$. △

Example 4. In the spirit of the Radon-Nikodym theorem, consider a (non-minimal) valence given by $\gamma(s) = k\phi(s)p(s)$ for some $k \geq 0$ and a function $\phi : S \to \mathbb{R}$ with $\int_S \phi dp = 0$. Then

$$V(f) = \int_S u(f)dp + \int_S u(f)\phi dp.$$

The willingness-to-bet on an event E is $\nu(E) = p(E) + \int_E \phi dp$ △

Remark 1. *In an SSEU representation, each state of the world is either "positive" or (weakly) "negative." Behaviorally, the sets of positive and negative states are defined by*

$$P = \{s \in S : x\{s\}y \succ y \text{ for some } x \succ y\},$$

$$N = \{s \in S : y \succcurlyeq x\{s\}y \text{ for some } x \succcurlyeq y\},$$

respectively.

It follows that any preference relation with an SSEU representation has an equivalent state-dependent representation of the form

$$V(f) = \sum_{s\in S} U(f(s), s),$$

where

$$U(x,s) = \begin{cases} (1+b)u(x)p^+(s) & \text{if } s \in P, \\ -bu(x)p^-(s) & \text{if } s \in N. \end{cases}$$

This is a very special type of state-dependent representation. It involves only two types of state-dependent utility, which differ in the sign of the marginal utility (one positive and the other negative) but which make the same relative comparisons between consequences. Indeed, consider two positive states $s, s' \in P$. The state-dependent utilities $U(\cdot, s)$ and $U(\cdot, s')$ are cardinally equivalent since they differ only by a positive constant. The same is true for any pair of negative states.

3 Axiomatic Characterization

This section introduces the axioms characterizing our SSEU model. Start with the standard Anscombe-Aumann axioms that characterize SEU.

Axiom 1 (Weak Order - WO)**.** $\succcurlyeq$ is complete and transitive. Moreover, there exist $f, g \in \mathcal{F}$ such that $f \succ g$.

Axiom 2 (Independence - I)**.** If $f, g, h \in \mathcal{F}$ and $\gamma \in (0, 1]$, $f \succcurlyeq g$ implies $\gamma f + (1-\gamma)h \succcurlyeq \gamma g + (1-\gamma)h$.

Axiom 3 (Archimedean - A)**.** If $f, g, h \in \mathcal{F}$ and $f \succ g \succ h$, there are $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.

Axiom 4 (Monotonicity - M)**.** For every $f, g \in \mathcal{F}$, $f(s) \succcurlyeq g(s)$ for every $s \in S \implies f \succcurlyeq g$.

Recalling the discussion in the Introduction, we weaken Monotonicity as follows[4]

Axiom 5 (Indifference Substitution - IS)**.** For every $f, g \in \mathcal{F}$, $f(s) \sim g(s)$ for every $s \in S \implies f \sim g$.

According to IS, two acts that yield equivalent payoffs in all states must be deemed indifferent.[5] To illustrate the weaker scope of IS compared to Monotonicity, consider two payoffs $x \succ y$. Under Monotonicity, it is necessary that $x \succcurlyeq xEy$ for all events E. However, IS allows for the possibility of $xEy \succ x$; i.e., the event E is "attractive" (see Section 3.2).

Replacing Monotonicity with Indifference Substitution characterizes our SSEU representation.

[4]One potential explanation for violation of Monotonicity is that this entails a subtle form of "state independence" of preferences, or weak separability, which may be overly restrictive. Monotonicity implies that, if $xEf \succ yEf$ for some event E and some act f, then $xFg \succcurlyeq yFg$ for all events F and acts g. In particular, if $x \succcurlyeq y$ then $xEf \succcurlyeq yEf$ for all events E and all acts f.

[5]This axiom appeared in Grant and Polak (2011), where it is called Substitution.

Theorem 2. *A binary relation $\succcurlyeq$ satisfies axioms WO, I, A, and IS if and only if there exists a non-constant affine function $u : X \to \mathbb{R}$ and a signed probability measure $\nu \in \Delta(S)$ such that (u, ν) is a SSEU representation of $\succcurlyeq$. Moreover, if (u', ν') is another SSEU representation of $\succcurlyeq$, then there exist $a, b \in \mathbb{R}$ with $a > 0$ and $u'(x) = au(x) + b$, and $\nu' = \nu$.*

By Proposition 1, the axioms in Theorem 2 are necessary and sufficient to obtain a Valence representation of $\succcurlyeq$. The next immediate result shows that Monotonicity implies the existence of a representation of $\succcurlyeq$ in which all events have "zero" valence.

Corollary 1. *A binary relation $\succcurlyeq$ satisfies axioms WO, I, A, and M if and only if it has a valence representation (u, p, γ) where $\gamma(E) = 0$ for all $E \subseteq S$.*

This corollary follows directly by taking $\gamma = \gamma^*$. By Monotonicity, the SSEU representation is an SEU representation, and $\nu = p^+$, implying $b = 0$.

3.1 Null Events

When a DM entertains signed probabilities, an event can be null but contain states that have non-zero probability. It therefore makes sense to posit that updating should in some way preserve beliefs over these internal states. We consider in particular the following distinct notions of null and non-null event.[6]

Definition 4. An event $E \subseteq S$ is $\succcurlyeq$**-null** if $xEy \sim y$ for some $x \not\sim y$; it is $\succcurlyeq$**-non-null** otherwise. An event $E \subseteq S$ is $\succcurlyeq$**-completely null** if every $F \subseteq E$ is $\succcurlyeq$-null. An event $E \subseteq S$ is $\succcurlyeq$**-classically null** if it there is no $F \subseteq E$ such that $xFy \succ y$ for some $x \succ y$.

The preceding definitions above coincide under SEU, but differ under SSEU. This can be seen from the following characterizations in terms of the minimal decomposition (p^*, γ^*).

Theorem 3. *Fix a preference relation $\succcurlyeq$ with an SSEU representation (u, ν) where u is non-constant, and let (p^*, γ^*) be the minimal decomposition of ν. An event E is $\succcurlyeq$-null if and only if $p^*(E) + \gamma(E) = 0$. An event is E $\succcurlyeq$-completely null if and only if $p^*(F) + \gamma^*(F) = 0$ for all $F \subseteq E$. An event E is $\succcurlyeq$-classically null if and only if $p^*(E) = 0$.*

3.2 Attractive and Aversive States

In our model, the DM attributes a valence to states, with some states viewed as "attractive" and other states viewed as "aversive." In this section, we formalize these notions more precisely in terms of the minimal valence.

[6] We are grateful to Miklós Pintér for suggesting the definition of a completely null event.

Definition 5. Fix a preference relation $\succcurlyeq$ with an SSEU representation (u, ν), where u is non-constant, and let (p^*, γ^*) be the minimal decomposition of ν. A state $s \in S$ is **attractive** (resp. **aversive**) if $\gamma^*(s) > 0$ (resp. $\gamma(s)^* < 0$).

A notable feature of our minimal valence concept is that it yields a simple behavioral characterization of aversive states, but only a sufficient condition for a state to be attractive.

Proposition 2. *Fix a preference relation $\succcurlyeq$ with an SSEU representation (u, ν), where u is non-constant, and let (p^*, γ^*) be the minimal decomposition of ν. A state $s \in S$ is aversive if and only if $y \succ x\{s\}y$ for some $x \succ y$. A state s is attractive if $x\{s\}y \succ x$ for some $x \succ y$.*

We see that aversive states are exactly those states on which the DM would not bet regardless of the payoffs. By contrast, if a state is such that the DM prefers to bet on it regardless of the payoffs, then that state is attractive.

3.3 Comparative Statics of Valence

In this section, we show that the absolute value of the valence function can be used as a measure to quantify the degree of non-classicality of a signed probability ν: the greater the value of $|\gamma^*|$, the more non-classical.

Consider two DM's with preferences $\succcurlyeq_1$ and $\succcurlyeq_2$. We show how the DM's can be ranked in terms of the extent to which their preferences depart from the classical framework. Define, for $i = 1, 2$,

$$P_i = \{s \in S : x\{s\}y \succ_i y \text{ for some } x \succ_i y\} \tag{3}$$

$$N_i = \{s \in S : y \succcurlyeq_i x\{s\}y \text{ for some } x \succcurlyeq_i y\}\,. \tag{4}$$

Definition 6. Say that DM 1 is **more non-classical** than DM 2 if, for all $f, g, h \in \mathcal{F}$ and $E \subseteq P_1 \cup P_2$,

$$fEh \succcurlyeq_1 gEh \implies fEh \succcurlyeq_2 gEh,$$

$$fEh \succ_1 gEh \implies fEh \succ_2 gEh,$$

and, for every $x, y, z \in X$ with $x \succ y$ and $E \subseteq N_1 \cup N_2$,

$$xEy \sim_1 z \implies xEy \succcurlyeq_2 z. \tag{5}$$

To understand this notion, observe that conditions (3) and (4) imply that–conditional upon the realization of a positive event–the two agents rank bets in the same way. It follows that, for example, if one positive event is considered more likely than another positive event according to agent 1, then it will be the case for agent 2 as well, and the same holds more the strict inequality. Furthermore, condition (5) indicates that agent 1 demands a higher compensation than agent

2 to participate in a bet involving a negative event. The next result shows that the absolute value of the valence function is a measure of non-classicality, therefore further motivating why our decomposition provides the most "classical" description of a DM's behavior.

Theorem 4. *The following conditions are equivalent*

(i) DM 1 *is more non-classical than DM* 2*;*

(ii) DM 1 *and DM* 2 *admit SSEU representations* (u, ν_1) *and* (u, ν_2) *with minimal decompositions* $\nu_1 = (p_1^*, \gamma_1^*)$ *and* $\nu_2 = (p_2^*, \gamma_2^*)$ *such that* $u_1 = u_2$*,* $p_1^* = p_2^*$*, and* $|\gamma_1^*| \geq |\gamma_2^*|$*.*

In addition, if DM 1 is more non-classical than DM 2, so that $|\gamma_1^*| \geq |\gamma_2^*|$, then $b_1 \geq b_2$. To see this, use $|b_1(p_1^+(s) - p_1^-(s))| \geq |b_2(p_2^+(s) - p_2^-(s))|$ for all $s \in S$, where $p_1 = p_2 = p$. Choosing a positive s implies $|b_1 p^+(s)| \geq |b_2 p^+(s)|$, or $b_1 \geq b_2$.

In the next example we consider simple parametric specifications of non-classicality.

Example 5 (Continuing Examples 1 and 3)**.** Suppose that DM 1 and DM 2 each have one aversive state s_1^* and s_2^*, respectively. Thus, $\gamma_1(s) = b_1(p_1^+(s) - \delta_{s_1^*})$ for some $s_1^* \in S$, and $\gamma_2(s) = b_2(p_2^+(s) - \delta_{s_1^*})$ for some $s_2^* \in S$, Then, if DM 1 is more non-classical than DM 2, then $p_1 = p_2 = p$, $u_1 = u_2$, and $|\gamma_1^*| \geq |\gamma_2^*|$. It follows that $s_1^* = s_2^* = s^*$. Using again $|\gamma_1| \geq |\gamma_2|$, we conclude that $|b_1(p(s) - \delta_{s^*})| \geq |b_2(p(s) - \delta_{s^*})|$, which is equivalent to $b_1 \geq b_2$.

Next, consider $\gamma(s) = b(p^+(s) - \frac{1}{|S|})$, so that

$$V(f) = \int_S u(f) dp^+ - b \left(\sum_{s \in S} (p^+(s) - \frac{1}{|S|}) u(f(s)) \right).$$

Again, we see that the degree of non-classicality is parameterized by b, where a larger b indicates a larger departure from classicality. △

4 Applications

In this section we provide some simple examples of the potential of event valence in explaining some types of behavior that are hard to reconcile with classical SEU. Unless otherwise noted, throughout the section we assume that the DM has SSEU preferences $\succcurlyeq$ with a valence representation (u, p, γ).

4.1 Hedging Aversion

Experimental evidence indicates reluctance on the part of sports fans to bet against a win by their favorite team (Morewedge et al., 2018; Kossuth et al., 2020; Donkor et al., 2023). This goes against standard SEU theory, which predicts that a well-designed hedge can mitigate disappointment generated by unfavorable outcomes.

In Morewedge et al. (2018), supporters of a sports team preferred a sure payment of \$0 to a bet paying \$20 if their preferred team lost a certain game. This preference was typically reversed when the bet was on a loss by a team different from the bettor's preferred one – even when the two teams had the same likelihood of losing. Rejecting a bet paying \$20 in favor of a sure payment of \$0 is a direct violation of Monotonicity (under the natural assumption that \$20 is preferred to \$0). Also, a difference in willingness-to-bet on two equally likely events is inconsistent with standard SEU theory, under which the likelihood of an event and the willingness-to-bet on it coincide.

To see that our SSEU theory can accommodate this pattern of behavior, let E be the event in which the DM's favorite team loses. Then $V(\$0) > V(\$20E\$0)$ whenever

$$u(0) > p(E)u(20) + (1 - p(E))u(0) + \gamma(E)(u(20) - u(0)),$$

which holds if and only

$$p(E) + \gamma(E) < 0.$$

Intuitively, if the event E has a sufficiently negative valence, the DM's willingness-to-bet on E becomes negative.

Now let F be the event that a team different from the DM's preferred one loses a game. Again, the DM chooses between a sure payment of \$0 and a bet paying \$20 if this team loses. Even if the two events E and F have the same likelihood, i.e., $p(E) = p(F)$, SSEU theory is consistent with both $V(\$0) > V(\$20E\$0)$ and $V(\$20F\$0) > V(\$0)$ whenever

$$-\gamma(E) > p(E) = p(F) > \gamma(F).$$

A particular case is $\gamma(F) = 0$, which says that the event F has no valence for the DM. Under SEU, we also have $\gamma(E) = 0$, so that this preference pattern is impossible.[7]

4.2 Conjunction Fallacy

Individuals exhibit the conjunction fallacy when they consider the probability of a conjunction of two events as larger than the probability of one of the constituent events (Tversky and Kahneman, 1982, 1983). Our SSEU model can accommodate instances of the conjunction fallacy, as the following immediate result illustrates:

Fact 1. *For any $E, F \subseteq S$, if $\gamma(E \cap F) - \gamma(F) = -\gamma(F \backslash E) > p(F) - p(E \cap F)$, then $\nu(E \cap F) > \nu(F)$.*

[7] Morewedge et al. (2018) found that as the financial gain increased (from \$20), fans were more likely to accept the bet against their team. Our theory cannot reproduce this effect. As we noted in the Introduction, we forego more detailed modeling in return for simplicity and, we believe, broader applicability.

For example, consider the following "update" of a famous experiment of Tversky and Kahneman (1983),[8] with four events associated with a match of Italian tennis star Jannik Sinner, as depicted in Table 1:

- E_1 = Sinner wins the match (rounded solid box)
- E_2 = Sinner loses the first set (solid box)
- E_3 = Sinner loses the first set but wins the match (dotted box)
- E_4 = Sinner wins the first set but loses the match (dashed box)

First set \ Match	win	lose
win	s_1	s_4
lose	s_3	s_2

Table 1: Events in the conjunction fallacy

In Tversky and Kahneman's experiment, subjects (on average) ranked the event E_1 more probable than event E_3, event E_3 more probable than E_2, and event E_4 as the least probable. A conjunction fallacy arises because event E_3 is the conjunction of events E_1 and E_2 (i.e., $E_3 = E_1 \cap E_2$). Its probability should not be larger than the probability of E_2.

Under our SSEU theory, a DM will exhibit the conjunction fallacy if $\nu(E_3) > \nu(E_2)$. This will happen if the valence of the state s_2, where Sinner loses both the first set and then the match, is sufficiently negative. For a Sinner fan, satisfaction of this condition could reflect the (anticipated) disappointment after a "total" loss.

Fact 2. *Suppose that $p(s_1) > -\gamma(s_1)$, $p(s_2) < -\gamma(s_2)$, and $p(s_4)+\gamma(s_4)$ is sufficiently small, then*

$$\nu(E_1) > \nu(E_3) > \nu(E_2) > \nu(E_4).$$

To see this, observe that by the first condition, $\nu(E_1) = p(s_1)+p(s_3)+\gamma(s_1)+\gamma(s_3) > p(s_3)+\gamma(s_3) = \nu(E_3)$. By the second condition, $\nu(E_3) = p(s_3)+\gamma(s_3) > p(s_3)+p(s_2)+\gamma(s_3)+\gamma(s_2) = \nu(E_2)$. Lastly, if $p(s_4)+\gamma(s_4) < \nu(E_2)$, we obtain the desired ranking.

As an example, the valence function defined by $\gamma(s_1) = -\gamma(s_2) > p(s_2)$ and $\gamma(s_3) = \gamma(s_4) = 0$ satisfies the conditions in Fact 2.

[8]The events in their experiment involved tennis great Björn Borg.

If, in addition to the conditions of Fact 2, we have $\gamma(E_1) = \gamma(E_1 \cap E_2) = \gamma(E_3)$, then $\nu(E_1) > \nu(E_1 \cap E_2) = \nu(E_3) > \nu(E_2)$. Next, consider the valence defined in Example 2, viz., $\gamma(s) = b(\delta_{s^*} - p^-(s))$. Let $s_1 = E_1 \backslash E_2$ be the attractive state. Then

$$\begin{aligned} \nu(E_1) &= 1 + b(1 - p^+(E_1)), \\ \nu(E_2) &= -bp^-(E_2), \\ \nu(E_3) &= \nu(E_1 \cap E_2) = -bp^-(E_1 \cap E_2). \end{aligned}$$

Thus, we find $\nu(E_1) > \nu(E_1 \cap E_2) = \nu(E_3) > \nu(E_2)$. If $p(s_4) = \nu(E_4)$ is smaller than $\nu(E_2)$, we obtain the desired ranking.

4.3 Coexistence of Insurance and Gambling

Standard expected utility theory with a concave or convex utility function struggles to explain why individuals simultaneously buy insurance and lottery tickets. The first purchase reflects risk aversion, while the second reflects risk seeking. Friedman and Savage (1948) suggests that risk attitudes can vary with wealth levels, so that insurance and gambling can coexist in a "sequential" purchases. We propose an alternative explanation based on valence that does not require sequentiality of purchases.[9]

An insurance policy covers a loss $\ell \geq 0$ if the event L occurs. The premium is denoted by q, and we assume that $0 \leq \nu(L) \leq 1$ and that $q = \nu(L)\ell$. A lottery pays $W \geq 0$ if the event E occurs and 0 otherwise. The price π of the lottery ticket is $0 \leq \pi \leq W$.

	L	L^c
f_1	$x - \ell$	x
f_2	$x - q$	$x - q$

Table 2: Acts f_1 and f_2

The DM compares four options $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$. The first option f_1 is to choose to purchase neither the insurance nor the lottery ticket. The first row of Table 2 shows the associated payoffs.

[9] If the price of a lottery ticket is small enough, the "sequential" explanation would require a large local change in the curvature of the utility function.

Its SSEU is[10]

$$V(f_1) = (p(L) + \gamma(L))u(x - \ell) + (1 - p(L) - \gamma(L))u(x).$$

The second option f_2 is to buy the insurance but not the lottery ticket. The second row of Table 2 shows the associated payoffs. Its SSEU is

$$V(f_2) = u(x - q).$$

The third option f_3 is to buy the insurance but not the lottery ticket. Table 3 shows the associated payoffs. Its SSEU is

$$V(f_3) = (p(L) + \gamma(L)) \left[(p(E) + \gamma(E))u(x - \pi + W - \ell) + (1 - p(E) - \gamma(E))u(x - \pi - \ell) \right] + (1 - p(L) - \gamma(L)) \left[(p(E) + \gamma(E))u(x - \pi + W) + (1 - p(E) - \gamma(E))u(x - \pi) \right].$$

Finally, with option f_4 the individual purchases both the insurance and the lottery ticket. Table 4 shows the associated payoffs. The SSEU of f_4 is

$$V(f_4) = (p(E) + \gamma(E))u(x - \pi + W - q) + (1 - p(E) - \gamma(E))u(x - \pi - q).$$

	L	L^c
E	$x - \pi + W - \ell$	$x - \pi + W$
E^c	$x - \pi - \ell$	$x - \pi$

Table 3: Act f_3

	L	L^c
E	$x - \pi + W - q$	$x - \pi + W - q$
E^c	$x - \pi - q$	$x - \pi - q$

Table 4: Act f_4

The coexistence of gambling and insurance requires $f_4 \succcurlyeq f_1$.

[10]In this section we make the assumption that X is the set of simple lotteries over $\mathbb{R}$, that is lotteries with finitely many possible outcomes over monetary amounts. Therefore, $u(x)$ is a standard von Neumann-Morgernstern expected utility. With a slight abuse of notation, we also use u to denote the Bernoulli utility of the von Neumann-Morgernstern representation. Monotonicity of u in this case means $\delta_k \succcurlyeq \delta_m$ when $k \geq m$. With this identification, a bet xEy pays $x = \delta_k$ if E occurs and $y = \delta_m$ if E^c occurs, and we write $u(x) = u(k)$, were the first u is the "expected utility" and the second u is the Bernoulli utility.

Fact 3. *Assume that the utility u is increasing and concave. If winning the lottery has a positive valence such that $p(E) + \gamma(E) \geq 1$, then there is coexistence of gambling and insurance.*

Actually, the conditions of Fact 3 imply $f_4 \succcurlyeq f_2 \succcurlyeq f_1$. Moreover, since u is concave, ceteris paribus, the DM always buys insurance, thus $f_4 \succcurlyeq f_3$. We see that if utility is concave and winning the lottery is a sufficiently attractive event —i.e., if $p(E) + \gamma(E) > 1$— an SSEU individual will buy both the insurance policy and the lottery ticket. Note that Fact 3 is true at any given level of wealth x: our analysis does not depend on wealth effects. Our mechanism operates by using the DM's valence to offset risk aversion. A sufficiently high valence creates risk-seeking behavior even for a DM with concave utility function u. The required condition can be stated in terms of what is known as Jensen's gap (for the probabilistic part p of the valence representation of ν)

$$u[p(E)x + (1 - p(E))y] - [p(E)u(x) + (1 - p(E))u(y)].$$

If this gap is less than the quantity $[u(x) - u(y)]\gamma(E)$, then, despite the concavity of u, the DM will choose the bet between x and y over its p-expectation defined as $p(E)x + (1 - p(E))y$.

Fact 4. *For any bet xEy with $x \succcurlyeq y$, if u is increasing and $p(E) + \gamma(E) \geq 1$, then $V(xEy) \geq u(p(E)x + (1 - p(E))y)$.*

The condition $p(E)+\gamma(E) \geq 1$ implies $\gamma(E) \geq 0$. Thus, if u is convex, the preceding inequality is trivially satisfied since the right-hand side is negative and the left-hand side is positive. The interesting case is when u is concave, and yet a "risk averse" DM can prefer a lottery to its expected value.

To link Fact 4 to the coexistence of gambling and insurance, note that if $p(E) + \gamma(E) \geq 1$, then, by Fact 3, $V(f_4) \geq u(x - q)$, but $u(x - q) \geq u(x - q - \pi + W) \geq u(x - q - \pi + p(E)W)$. Therefore, $V(f_4) \geq u(x - q - \pi + p(E)W)$, and $x - q - \pi + p(E)W$ is the expected value of the lottery under p in the presence of insurance.

4.4 Dominated Choice in Strategy-Proof Mechanisms

Recent empirical evidence suggests that some individuals choose dominated (in the sense of first-order stochastic dominance) strategies in strategy-proof mechanisms, such as the Deferred Acceptance (DA) mechanism much studied in the context of school or college choice (Hassidim et al., 2016; Shorrer and Sóvágó, 2023; Dreyfuss et al., 2022).

Flipping and truncation are two examples of dominated strategy choice. Under flipping, an individual submits a ranking that reverses the "obvious" order of two alternatives—e.g., ranking a school choice that comes with a fellowship below the same choice without a fellowship. Under truncation, an individual submits a restricted ranking that omits some schools. Both types of behavior are inconsistent with standard SEU, which respects first-order stochastic dominance, but they are consistent with our SSEU theory.

Consider two schools σ_1 and σ_2. We let $u(x_1) = m_1 > 0$ denote the utility of being matched with school σ_1, $u(x_2) = m_2$ the utility of being matched with school σ_2, and $u(x_0) = 0$ the utility of not being matched. Clearly, being matched with both σ_1 and σ_2 is not possible under the DA. There are four possible rankings which we denote by $\sigma_1 \triangleright \sigma_2$, $\sigma_2 \triangleright \sigma_1$, σ_1, and σ_2. The first two rankings are complete, while the third and fourth are truncations.

Let E_1 be the event of being matched with school σ_1, E_2 the event of being matched with school σ_2, and $E_{\neg 1,2}$ the event of being matched with school σ_2 conditional on not being matched with school σ_1. Submitting the ranking $\sigma_1 \triangleright \sigma_2$ generates the act that yields x_1 if E_1 occurs, x_2 if $E_{\neg 1,2}$ occurs, and x_0 otherwise. The analogous act corresponds to submitting $\sigma_2 \triangleright \sigma_1$. Following Dreyfuss et al. (2022), we assume for simplicity that the probability assessment of being matched to school σ_2 conditional on not being matched to school σ_1 can be written as $p(E_{\neg 1,2}) = (1 - p(E_1))p(E_2)$, and similarly for the event of being matched to σ_1 conditional on not being matched to σ_2.

Submitting a ranking that contains only σ_1 generates the act $x_1 E_1 x_0$ that yields x_1 if E_1 occurs and x_0 otherwise. Likewise, submitting a ranking that contains only σ_2 generates the act $x_2 E_2 x_0$ that yields x_1 if E_2 occurs and x_0 otherwise.

Suppose that an individual ranks school σ_1 higher than σ_2, i.e., $m_1 > m_2$. Flipping means that the individual submits the ranking $\sigma_2 \triangleright \sigma_1$. The SSEU of submitting a faithful ranking is $\nu(E_1)m_1 + \nu(E_{\neg 1,2})m_2$, whereas the SSEU of the flipped ranking is $\nu(E_2)m_2 + \nu(E_{\neg 2,1})m_1$. Submitting a flipped ranking is preferred if

$$p(E_1)p(E_2)(m_1 - m_2) < (\gamma(E_{\neg 2,1}) - \gamma(E_1))m_1 - (\gamma(E_{\neg 1,2}) - \gamma(E_2))m_2.$$

The left-hand side of this inequality is always positive, and so the right-hand side has to be sufficiently positive if flipping is to be optimal. For example, suppose that all events in the inequality have valence 0 except for the aversive event $E_{\neg 1,2}$, corresponding to being matched with the less preferred school σ_2 over the more preferred school σ_1. The inequality then becomes

$$p(E_1)p(E_2)(m_1 - m_2) < -m_2\gamma(E_{\neg 1,2}),$$

which will hold if $\gamma(E_{\neg 1,2})$ is sufficiently smaller than 0. Clearly, since under SEU we have $\gamma \equiv 0$, this inequality is never satisfied.

Truncation, by submitting just σ_1 rather than the ranking $\sigma_1 \triangleright \sigma_2$, is preferred if

$$\nu(E_1)m_1 > \nu(E_1)m_1 + \nu(E_{\neg 1,2})m_2,$$

which will hold whenever $\gamma(E_{\neg 1,2}) < 0$ is small enough, i.e., there is enough aversion to be rejected by the more preferred school.

Our explanation of flipping and truncation behavior in the DA mechanism differs from the explanation given by Dreyfuss et al. (2022). These authors suggest that "an applicant who is likely to get matched with a school will feel a loss when matched with any other school (even a better one); this can create attachment to the high-probability school—an endowment effect for schools." From this idea they conclude that flipping and even truncation might be observed in highly loss-averse individuals. Our explanation is not dependent on a (high) likelihood of being matched with a particular school, but on the aversiveness of not being matched with a preferred school.

5 Extension: Signed Invariant Biseparable Preferences

We motivated our development of SSEU via the idea of events that have positive or negative valence for the DM. We then found that the valence function associated with a signed probability measure necessarily satisfies the two properties of (disjoint) additivity and null-additivity with respect to complements that comprised Definition 2. Neither of these properties strikes us as essential to our intuition that events may carry a valence. Our theory imposes a further additivity via the signed probability measure ν, which comes from the fact that we assumed the full force of the Independence axiom. Of course, Independence is well known to be violated in the presence of factors such as ambiguity (Ellsberg, 1961) that may affect a DM's willingness to bet. In this section we extend SSEU to allow for a non-additive willingness to bet, by relaxing Independence. As a by-product, we will end up with fewer constraints on the structure of the (generalized) valence for the DM.

A function $\rho : 2^S \to \mathbb{R}$ satisfying $\rho(\emptyset) = 0$ and $\rho(S) = 1$ is called a *signed capacity*.[11] If, in addition, ρ is monotone, that is, if $A \subseteq B$ implies $\rho(B) \geq \rho(A)$, then it is called a *capacity* (Choquet, 1953).

Definition 7. We say that (V, ρ) is a **signed invariant biseparable** (**SIBS**) **representation** of $\succcurlyeq$ where $V : \mathcal{F} \to \mathbb{R}$ and ρ is a signed capacity on 2^S such that

1. for all $x \succcurlyeq y$ in X and $E \subseteq S$, we have

$$V(xEy) = V(x)\rho(E) + V(y)(1 - \rho(E));$$

2. for all $f \in \mathcal{F}$, $x \in X$, and $\alpha \in [0, 1]$, we have

$$V(\alpha f + (1 - \alpha)x) = \alpha V(f) + (1 - \alpha)V(x).$$

[11] Signed capacities correspond to the notion of a game in cooperative game theory; see, e.g., Aumann and Shapley (1974).

The SIBS representation generalizes the invariant biseparable representation of Ghirardato and Marinacci (2001) by allowing for a non-monotone willingness to bet.

The key modification in our new axiomatization is the replacement of Independence with a substantially less restrictive condition. The axiom we propose generalizes Independence since in imposing constraints on preferences only when comparing mixtures that involve a constant act (Gilboa and Schmeidler, 1989).

Axiom 6 (Certainty Independence - CI)**.** For all acts $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1]$, if $f \succ g$ then $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$.

Theorem 5. *A binary relation $\succcurlyeq$ satisfies axioms WO, A, IS, and CI if and only if there is a non-constant affine function $V : \mathcal{F} \to \mathbb{R}$ and a signed capacity $\rho : 2^S \to \mathbb{R}$ such that (V, ρ) is an SIBS representation of $\succcurlyeq$. Moreover, if (u', ρ') is another SIBS representation of $\succcurlyeq$, then there exist $a, b \in \mathbb{R}$ with $a > 0$ and $u'(x) = au(x) + b$, and $\rho' = \rho$.*

Paralleling our earlier decomposition of a signed probability measure ν, a signed capacity ρ can be decomposed into a capacity and a function reflecting valence. That is, we can write

$$\rho = \mu + \Gamma,$$

where $\mu : 2^S \to [0,1]$ is a capacity and the function $\Gamma : 2^S \to \mathbb{R}$, which we call a *generalized valence*, satisfies only $\Gamma(\emptyset) = \Gamma(S) = 0$.

Definition 8. We say that (V, μ, Γ) is a **generalized valence (GV) representation of** $\succcurlyeq$ if $V : \mathcal{F} \to \mathbb{R}$, μ is a capacity on 2^S, and Γ is a generalized valence such that:

1. for all $x \succcurlyeq y$ in X, and $E \subseteq S$, we have

$$V(xEy) = V(x)\mu(E) + V(x)\Gamma(E) + V(y)(1-\mu(A)) + V(y)(1-\Gamma(A));$$

2. for all $f \in \mathcal{F}$, $x \in X$, and $\alpha \in [0,1]$, we have

$$V(\alpha f + (1-\alpha)x) = \alpha V(f) + (1-\alpha)V(x).$$

Proposition 3. *The preferences $\succcurlyeq$ have an SIBS representation if and only if they admit a GV representation.*

As discussed in Ghirardato and Marinacci (2001), the Choquet expected utility model (Schmeidler, 1989) corresponds to the particular case of the invariant biseparable representation (and thus of SIBS) where ρ is a capacity and the functional V is a Choquet integral (Choquet, 1953) with respect to ρ. In our setting with valence, the classical Choquet Expected Utility (CEU) model thus generalizes as follows. A **signed Choquet expected utility** (**SCEU**) representation of $\succsim$ is a pair (u, ρ) in which $u : X \to \mathbb{R}$ is an affine function and $\rho : 2^S \to \mathbb{R}$ is a signed capacity, and

$$f \succsim g \iff \int_S u(f) d\rho \geq \int_S u(g) d\rho.$$

An SCEU representation can be decomposed as

$$V(f) = \int_S u(f) d\mu + \int_S u(f) d\Gamma,$$

where μ is a positive capacity with $\mu(S) = 1$ and Γ is a generalized valence.

The SCEU model (and the SIBS model) can reproduce the preference pattern observed experimentally by Schneider and Schonger (2019). They found evidence for violations of the following consequence of Monotonicity: If $xEz \succ yEz$ for some $x, y, z \in X$ and an event E, then $xEz' \succsim yEz'$ for all $z' \in X$.

Example 6. Fix payoffs $x, y \in X$ with $x \succ y$ and consider an event E and a capacity ρ with $\rho(E) > 0$. Given $z \in X$ with $y \succsim z$, we have

$$V(xEz) = \rho(E)u(x) + (1 - \rho(E))u(z) > \rho(E)u(y) + (1 - \rho(E))u(z) = V(yEz).$$

That is, $xEz \succ yEz$. Now suppose that $1 - \rho(E^c) < 0$. Then, given $z' \succsim x$, we have

$$V(yEz') = \rho(E^c)u(z') + (1 - \rho(E^c))u(y) > \rho(E^c)u(z') + (1 - \rho(E^c))u(x) = V(xEz').$$

That is, $yEz' \succ xEz'$. Notice that if the capacity ρ is a signed measure, then the conditions $\rho(E) > 0$ and $1 - \rho(E^c) < 0$ cannot be simultaneously satisfied. Also, the capacity ρ is not convex, since $\rho(E) + \rho(E^c) > 1$. $\triangle$

6 Discussion

We conclude with some conceptual remarks, further comments on the literature, and a connections to the analysis of physical systems.

a. Dutch-Book Objection There is a possible objection to our theory, which is that a DM following SSEU might be susceptible to a "Dutch Book," and a consequent money pump (de Finetti, 1937). If the DM's willingness-to-bet on event E is $\$\nu(E)$, where $\nu(E) > 1$, then a bettor who does not share this assessment will be able to obtain a sure win, by selling the DM a bet with prize \$1 if E obtains and \$0 otherwise—at a price of $\$(\nu(E) - \epsilon)$ (where $\epsilon < \nu(E) - 1$). But, while this might be feasible as a one-time transaction, it is not clear to us that it will be repeatable. After the first transaction, the DM might well change valence so that $\nu(E) \leq 1$ for subsequent bets. Money pumps can occur only in the presence of unchanging preferences over the course of the pump, and we see no reason to assume that valence might not vary with the

DM's betting history. Even for the one-time bet, the sure loss looks troublesome only from an outsider's perspective (assuming the outsider follows classical SEU). From the DM's perspective, the positive valence attached to E implies a (non-monetary) welfare gain from betting on E.

b. Related Literature There are other papers in decision theory where signed probabilities arise, although interpreted differently from here or left uninterpreted. Dekel et al. (2001, 2007) study preference over menus of lotteries and provide an axiomatization of an expected utility representation with respect to a signed probability measure on a subjective state space. In this framework, "positive" states represent potential future normative preferences, while "negative" states model possible future temptations that could affect preferences. De Waegenaere and Wakker (2001) demonstrate how signed Choquet integrals can be applied to intertemporal preferences. In particular, by allowing violations of monotonicity, their model encompasses a DM who prefers an increasing sequence of consumption over a decreasing sequence, even if the latter involves a higher consumption level at each point in time. Perea (2022) develops a model motivated by game theory which axiomatizes a player's conditional preference relation over acts, where the conditioning is on different probability measures ("beliefs") the DM might hold over choices made by other players. When extended to signed probabilities, the axioms yield a utility (payoff) function for the player, justifying the conventional specification of a payoff matrix for a game. Ke and Zhao (2023) provide new models for decision making under ambiguity, one of which – called "cautiously optimistic linear utility" – features a collection of sets of possibly signed (subjective) measures over states.

c. Physical Systems We next mention a connection between our decision theory and quantum-mechanical physical systems. While quantum mechanics (QM) is usually formulated in terms of Hilbert spaces and Hermitian operators (Sakurai and Napolitano, 2020, is a standard introduction), there is a fully equivalent formulation in terms of phase space (analogous to the state space of decision theory) and signed probabilities. This equivalence was first formulated by Wigner (1932), and the use of signed probabilities in QM was promoted by Dirac (1942) and Feynman (1987). Today, phase-space QM is routinely employed in e.g. quantum optics (Kenfack and Życzkowski, 2004). Since the probabilities in QM are objective, only unobserved events can receive probability outside the interval $[0, 1]$. But, even though the appearance of non-classical probability is restricted in this fashion, this is sufficient to allow observed behavior, such as quantum entanglement (see Horodecki et al., 2009, for a survey), that is impossible in the classical physical world.

We believe that our SSEU theory may find useful applications to QM. The idea is that a DM's signed probabilities might reflect the objectively-given signed probabilities in QM. There will be subtleties. Thus, we already noted the issue of unobservability of certain events in a quantum system, so that the question of verifiability or not of certain bets made by (or between) DM's interacting with such a system needs thought. At the calculational level, our novel decomposition

$\nu = p + \gamma$ of signed probabilities suggests a new measure of the degree of non-classicality of a quantum system based on the total variation of γ, that is, $\|\gamma\|_v = \sum_{s\in S} |\gamma(s)|$. See Camillo and Cervantes (2024) for a review of existing measures of non-classicality (called contextuality) in QM.[12]

A Appendix: Proofs

A.1 Proof of Proposition 1

The first result provides an intermediate step in the proof of Proposition 1.

Proposition 4. *Assume that $\succcurlyeq$ admits the SSEU representation (u, ν). Then there exist $p^+, p^- \in \Delta^+(S)$, $b \geq 0$ such that*

$$\nu(s) = (1 + b)p^+(s) - bp^-(s),$$

for every $s \in S$, so that for every $f \in \mathcal{F}$

$$\int_S u(f)d\nu = (1 + b)\int_S u(f)dp^+ - b\int_S u(f)dp^-.$$

Moreover, $\succcurlyeq$ satisfies axiom 4 if and only if $b = 0$.

Proof. Let $a = \nu^+(S)$ and $b = \nu^-(S)$. Clearly $a \geq 0$ and $b \geq 0$ and $\nu(S) = 1$ excludes that $a = b = 0$. If $b = 0$, then $\nu = \nu^+$ and $a = 1$. Moreover, if $a < 1$ then $1 = \nu(S) = a - b$ would imply $b \leq 0$ that contradicts the fact that ν^- is a positive measure, therefore $a \geq 1$. By setting $p^+(s) = \frac{\nu^+(s)}{a}$ and $p^- = \frac{\nu^-(s)}{b}$ (if $b > 0$, otherwise $p^+ = \nu^+ = \nu$) for every $s \in S$, we have that $p^+, p^- \in \Delta^+(S)$. Moreover, we have

$$\nu(s) = \nu^+(s) - \nu^-(s) = \frac{a\nu^+(s)}{a} - \frac{b\nu^-(s)}{b} = ap^+(s) - bp^-(s) \text{ for every } s \in S.$$

Now let

$$N = \{s \in S : x\{s\}y \succ y \text{ for some } y \succ x\},$$

and observe that $s \in N$ if and only if $\nu(s) < 0$. If $\succcurlyeq$ satisfies monotonicity, then $N = \emptyset$, and since $b = \nu^-(S) = \nu^-(S \cap N) = \nu^-(\emptyset) = 0$. Hence $b = 0$ as desired. If $b = 0$, then $\succcurlyeq$ satisfies monotonicity. □

[12] There is a literature (e.g., La Mura, 2009; Busemeyer and Bruza, 2014) that employs aspects of the QM formalism to build non-classical decision theories in order to explain behavioral anomalies – where these anomalies are posed in the classical realm, as usual. Our remarks about a connection to QM go in the opposite direction, since our DM is fully classical in the physical sense. The point we make here is that one reason for a DM to entertain signed probabilities is when interacting with a (microscopic) quantum system.

Proof of Proposition 1. By defining $\nu(s) = p(s) + \gamma(s)$, a valence representation can be rewritten as $V(f) = \int_S u(f)d(p+\gamma) = \int_S u(f)d\nu$. The set function ν is such that, for any $E, F \subseteq S$ with $E \cap F = \emptyset$, $\nu(E \cup F) = p(E \cup F) + \gamma(E \cup F) = p(E) + p(F) + \gamma(E) + \gamma(F) = \nu(E) + \nu(F)$, where the second equality follows from p being a probability and property 1 of γ in Definition 2. Lastly, point 2 in Definition 2 implies that $\nu(S) = p(S) + \gamma(S) = 1 + 0 = 1$, so that $\nu \in \Delta(S)$.

For the opposite implication, suppose $\succcurlyeq$ has a SSEU representation (u, ν). By Proposition 4, there is $b \geq 0$, and $p^+, p^- \in \Delta^+(S)$ such that $\nu(s) = (1+b)p^+(s) - bp^-(s)$. The valence representation follows by defining $p(s) = p^+(s)$ and $\gamma(s) = b(p^+(s) - p^-(s))$. □

A.2 Proof of Theorem 1.

Given a signed measure ν with $\nu(S) = 1$, it is standard to prove that $\|\nu\|_v = \sum_{s\in S} |\nu(s)| = \sup_{E\subseteq S}[\nu(E) - \nu(E^c)]$. First, we define the set

$$A_{\nu^+} = \left\{p \in \Delta^+(S) : p \leq \nu^+\right\}.$$

Second, we show that $A_{\nu^+} = \text{argmin}_{q\in\Delta^+(S)} \|\nu - q\|_v$. To prove this inequality, we show that $\|\nu - q\|_v \geq 2b$ for all $q \in \Delta^+(S)$. Indeed,

$$\begin{aligned}\|\nu - q\|_v =& \sup_{E\subseteq S}[\nu(E) - p(E) - \nu(E^c) + p(E^c)] \\ =& 2\sup_{E\subseteq S}[\nu(E) - p(E)].\end{aligned}$$

Taking $E = P$, we get $\|\nu - q\|_v \geq 2(\nu(P) - q(P)) \geq 2(1 + b - 1) = 2b$. At the same time,

$$\|\nu - q\|_v = \sum_{s\in P} |\nu(s) - q(s)| + \sum_{s\in N} |\nu(s) - q(s)|.$$

Therefore,

$$\begin{aligned}\|\nu - q\|_v =& \sum_{s\in P} |\nu(s) - q(s)| + \sum_{s\in N} (\nu^-(s) + q(s)) \\ =& \sum_{s\in P} |\nu(s) - q(s)| + b + q(N) = \sum_{s\in P} |\nu(s) - q(s)| + 1 - q(P) \\ =& 1 + b + \sum_{s\in P}[|\nu(s) - q(s)| - q(s)].\end{aligned}$$

If $q \in A_{\nu^+}$, then $\|\nu - q\|_v = 1 + b + \sum_{s\in P}[\nu(s) - 2q(s)] = 1 + b + 1 + b - 2 = 2b$, thus $A_{\nu^+} \subseteq \text{argmin}_{q\in\Delta^+(S)} \|\nu - q\|_v$. For the converse inclusion. Take $q \in \Delta^+(S)$ such that $\|\nu - q\|_v = 2b$. Then, it must be that $q(P) = 1$, since otherwise $\|\nu - q\|_v = 2 + 2b - 2q(P) > 2b$, which is impossible. Therefore, $\sum_{s\in P}[|\nu(s) - q(s)| - q(s)] = 1 + b - 2 = \sum_{s\in P}[\nu(s) - 2q(s)]$, which implies

$$\sum_{s\in P}[|\nu(s) - q(s)|] = \sum_{s\in P}[\nu(s) - q(s)]$$

or

$$2\sum_{s\in P}(\nu(s) - q(s))^- = 0,$$

so that $q \in A_{\nu^+}$. Lastly, if (p^*, γ^*) is the minimal decomposition of ν, $p^* = \frac{\nu^+}{1+b} \leq \nu^+$, thus $p^* \in A_{\nu^+}$.

Suppose that p satisfies (2) and preserves willingness-to-bet ratios $\nu(s)/\nu(s') = p(s)/p(s')$ for all $s, s' \in P$. If $|P| = 1$, the result is trivially true. Assume that $|P| > 1$. Since p^* preserves likelihood ratios as well, it follows that $p(s)/p(s') = p^*(s)/p^*(s')$, so that $p^*(s) = p(s)\,(p^*(s')/p(s'))$. Summing over P, we obtain $1 = \sum_{s\in P} p^*(s) = (p^*(s')/p(s')) \sum_{s\in P} p(s)$ so that $p^*(s') = p(s')$. Since s, s' were arbitrary, the result follows.

A.3 Proof of Theorem 2

Several steps of this proof are standard, but we report them for completeness. We denote with B the set of functions from S to $\mathbb{R}$, and given $K \subseteq \mathbb{R}$ we denote with $B(K)$ the set of functions from S to K.

Since $\mathcal{F}$ is mixture set and because $\succcurlyeq$ satisfies axioms 1-3, by Theorem 8 in Herstein and Milnor (1953) there exists $V : \mathcal{F} \to \mathbb{R}$ that represents $\succcurlyeq$ such that

$$V\left(\alpha f{+}(1-\alpha)g\right) = \alpha V(f) + (1-\alpha)V(g),$$

for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$. Moreover, V is unique up to positive affine transformations. Further, observe that we can find $\bar{x}, \underline{x} \in X$ such that $V(\bar{x}) = -V(\underline{x}) = 1$. To see this, notice that there exist $\bar{x}, \underline{x} \in X$ such that $\bar{x} \succ \underline{x}$. Indeed, if $x \sim x'$ for every $x, x' \in X$, then it would follow by axiom 5 that $f \sim g$ for every $f, g \in \mathcal{F}$, contradicting axiom 1. Therefore without loss of generality we can take V such that $V(\bar{x}) = 1$ $V(\underline{x}) = -1$.

Define $u : X \to \mathbb{R}$ by $u(x) = V(x)$ for every $x \in X$. Let $K := u(X)$. Then K is convex by the affinity of V and without loss of generality satisfies $[-1, 1] \subseteq K$ by the previous paragraph.

Define a function $U : \mathcal{F} \to B(K)$ as follows: for every $f \in \mathcal{F}$ and $s \in S$

$$U(f)(s) := u(f(s)).$$

Then, U is surjective since $K = u(X)$ and U satisfies that $U(f) = U(g) \Rightarrow f \sim g$ by axiom 5. Therefore, U is bijective. Furthermore, the affinity of V implies that for all α and $f, g \in \mathcal{F}$

$$U(\alpha f + (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g).$$

Now, define a functional I on $B(K)$ by

$$I(\phi) = V\left(U^{-1}(a)\right)$$

for all $\phi \in B(K)$. Clearly, it holds that for all $f \in \mathcal{F}$

$$I(U(f)) = I(u \circ f) = V(f).$$

Moreover, $I(x1_S) = x$ for every $x \in K$.

Now let $\phi, \psi \in B(K)$ and let $\alpha \in [0,1]$. Also, let $f, g \in \mathcal{F}$ be such that $U(f) = a$ and $U(g) = b$. Since U is surjective, such f and g exist. Then, we obtain

$$\begin{aligned} I(\alpha a + (1-\alpha)b) &= V\left(U^{-1}(\alpha a + (1-\alpha)b)\right) \\ &= V\left(U^{-1}(\alpha U(f) + (1-\alpha)U(g))\right) \\ &= V\left(U^{-1}(U(\alpha f + (1-\alpha)g))\right) \\ &= V(\alpha f + (1-\alpha)g) \\ &= \alpha V(f) + (1-\alpha)V(g) \\ &= \alpha I(a) + (1-\alpha)I(b). \end{aligned}$$

It follows that I is positively homogeneous, i.e. for every $\alpha > 0$ and $\phi \in B(K)$ such that $\alpha\phi \in B(K)$ it holds $I(\alpha\phi) = \alpha I(\phi)$. Moreover, I is additive, that is for every $\phi, \psi \in B(K)$ such that $\phi + \psi \in B(K)$ it holds $I(\phi + \psi) = I(\phi) + I(\psi)$. To see this point, observe that by positive homogeneity we obtain $I(\phi + \psi) = I\left(2\left(\frac{\phi}{2} + \frac{\psi}{2}\right)\right) = 2I\left(\frac{\phi}{2} + \frac{\psi}{2}\right) = 2\left(\frac{I(\phi)}{2} + \frac{I(\psi)}{2}\right) = I(\phi) + I(\psi)$.

We can now extend I to B as follows. For every $\phi \in B$ there exists $\psi \in B(K)$ and $\alpha \geq 1$ such that $\psi = \alpha\phi$. We can therefore define $\bar{I} : B \to \mathbb{R}$ by

$$\bar{I}(\phi) = \alpha I(\psi).$$

Observe that it is now immediate to satisfies $\bar{I}$ satisfies for every $\alpha \in \mathbb{R}$ and $\phi, \psi \in B$

1. $\bar{I}(\phi + \psi) = \bar{I}(\phi) + \bar{I}(\psi)$;

2. $\bar{I}(\alpha\phi) = \alpha\bar{I}(\phi)$.

Now by the Riesz representation theorem (e.g., see Theorem 6.45 in Axler (1997)) there exists a (unique) $\nu : S \to \mathbb{R}$ such that for every $\phi \in B$

$$\bar{I}(\phi) = \int \phi d\nu = \sum_{s \in S} \phi(s)\nu(s).$$

We can therefore conclude that

$$f \succcurlyeq g \iff V(f) \geq V(g) \iff I(u(f)) \geq I(u(g)) \iff \int u(f) d\nu \geq \int u(g) d\nu.$$

Note that we can assume that $\nu(S) = \sum_{s \in S} \nu(s) = 1$. Indeed, it must be that $\sum_{s \in S} \nu(s) = \nu(S) \neq 0$ since $I(1_S) = 1$. Further, if $\sum_{s \in S} \nu(s) = \nu(S) \neq 1$, then letting let $\tilde{\nu}(s) = \frac{\nu(s)}{\nu(S)}$ and $\tilde{u} = \nu(S)u$, so that

$$V(f) = I(u(f)) = \int u(f)\nu = \int \tilde{u}(f)\tilde{\nu},$$

so that the claim is satisfied.

Finally, if there exists $\nu' \in \Delta(S)$ and an affine function $u' : X \to \mathbb{R}$ such that (u', ν') represents $\succcurlyeq$, then because V is unique up to affine transformations there exists $a > 0$ and $b \in \mathbb{R}$ such that

$$\int u'(f)d\nu' = a \int u(f)d\nu + b$$

for every $f \in \mathcal{F}$ from which we obtain

$$u'(x) = au(x) + b,$$

for every $x \in X$. We can therefore conclude that $\nu' = \nu$ as desired.

A.4 Proof of Theorem 3

It is easy to see that $E \subseteq S$ is $\succcurlyeq$-null iff $\nu(E) = p^*(E) + \gamma^*(E) = 0$. This immediately implies the characterization of $\succcurlyeq$-completely null events, by additivity of ν.

Next, using the definitions of p^+ and p^- from Proposition 4, we see that for every $s \in S$ we have $\nu(s) = 0$ iff $p^*(s) = p^+(s) = 0$ and $p^-(s) = 0$. (For the "only if" direction, notice that by definition $\nu(s) = 0$ implies $\nu^+(s) = 0 = \nu^-(s)$, which immediately implies $p^+(s) = 0$ and $p^-(s) = 0$.) Analogously, $\nu(s) > 0$ iff $p^*(s) = p^+(s) > 0$ and $p^-(s) = 0$, and $\nu(s) < 0$ iff $p^*(s) = p^+(s) = 0$ and $p^-(s) > 0$. If E is $\succcurlyeq$-classically null, it cannot contain any state s such that $x\{s\}y \succ y$ for some $x \succ y$ (which would imply $\nu(s) > 0$), hence $p^*(E) = 0$. Conversely, if $p^*(E) = 0$, E cannot clearly contain any event F such that $p^*(F) > 0$, and so it is $\succcurlyeq$-classically null.

A.5 Proof of Fact 3

According to SSEU, the difference between $V(f_2)$ and $V(f_1)$ is

$$u(x - q) - \nu(L)u(x - \ell) - (1 - \nu(L))u(x).$$

Since $0 \le \nu(L) \le 1$, Jensen's inequality yields

$$V(f_2) - V(f_1) = u(x - q) - \nu(L)u(x - \ell) - (1 - \nu(L))u(x) \ge$$
$$u(x - q) - u(x - \nu(L)\ell) = 0.$$

Thus, the DM buys the insurance policy. The SSEU difference between f_4 and f_2 is

$$V(f_4) - V(f_2) = \nu(E)u(x - q - \pi + W) + (1 - \nu(E))u(x - q - \pi) - u(x - q).$$

By assumption, $\nu(E) = p(E) + \gamma(E) = 1 + \epsilon$ for some $\epsilon \ge 0$. Also, $u(x - q - \pi + W) - u(x - q) \ge 0$ since u is increasing and $W \ge \pi$. Similarly, $u(x - q - \pi + W) - u(x - q - \pi) \ge 0$. Therefore

$$u(x - q - \pi + W) + \epsilon(u(x - q - \pi + W) - u(x - q - \pi)) - u(x - q) \ge 0,$$

because $W \ge \pi$, and thus $V(f_4) \succcurlyeq V(f_2)$. We conclude that $V(f_4) \succcurlyeq V(f_2) \succcurlyeq V(f_1)$.

A.6 Proof of Fact 4

If $p(E)+\gamma(E)\geq 1$, then $\gamma(E)\geq 1-p(E)$. Since $u(x)\geq u(y)$, we have $\gamma(E)[u(x)-u(y)]\geq (1-p(E))[u(x)-u(y)]$. Adding and subtracting $p(E)u(x)$ on the right-hand side implies

$$\gamma(E)[u(x)-u(y)]\geq u(x)-[p(E)u(x)+(1-p(E))u(y)].$$

Since u is increasing, we can write $u(x)\geq u(p(E)x+(1-p(E))y)$,and thus

$$\gamma(E)[u(x)-u(y)]\geq u(p(E)x+(1-p(E))y)-[p(E)u(x)+(1-p(E))u(y)],$$

as required.

A.7 Proof of Theorem 4

$(i)\implies(ii)$: First observe that 1 and 2 admit SSEU representations given by $(u,\nu_1),(u,\nu_2)$ such that $u_1=u_2$. To see this, observe that since 1 is more classical than 2, taking $f=x, g=y$ it follows that

$$x\succcurlyeq_1 y \iff xEh\succcurlyeq_1 yEh \iff xEh\succcurlyeq_2 yEh \iff x\succcurlyeq_2 y,$$

for all $x,y\in X$. Further observe that $P_1=P_2:=P$. To see this, take $s\in P_1$. Then it must be that $x\{s\}y\succ_1 y$. Since 1 is more classical than 2, it follows that letting $x\{s\}y\succ_2 y$, which implies $s\in P_2$, so that $P_1\subseteq P_2$. One can show in the same way that $P_2\subseteq P_1$. It follows that $N_1=N_2:=N$.

Now define $\succcurlyeq_i'$, $i=1,2$ on X^P as follows

$$f\succcurlyeq_i' g \iff \text{there exists } h\in\mathcal{F} \text{ such that } fPh\succcurlyeq_i gPh,$$

for every $f,g\in X^P$. Since 1 is more classical than 2, we obtain that

$$f\succcurlyeq_1' g \iff f\succcurlyeq_2' g,$$

it follows that

$$p_1^*(s)=\frac{\nu_1(s)}{\nu_1(P)}=\frac{\nu_2(s)}{\nu_2(P)}=p_2^*(s)=p^*(s) \text{ for every } s\in P.$$

Finally, given any $s\in N$, choose x,y,z such that $x\{s\}y\sim_1 z$.[13] Since 1 is more classical than 2 it follows that

$$\nu_1(s)=\frac{u(y)-u(z)}{u(x)}\leq \nu_2(s),$$

which implies that $\gamma_1^*(s)\leq\gamma_2^*(s)$ whenever $s\in N$. When $s\in P$, $\gamma_1^*(s)=b_1p^*(s)\geq b_2p^*(s)=\gamma_2^*(s)$. We can therefore conclude that $|\gamma_1^*|\geq|\gamma_2^*|$.

$(ii)\implies(i)$: Assume that $p_1^*=p_2^*=p^*$. Take $f,g,h\in\mathcal{F}$ and $E\subseteq P_1\cup P_2$. We have

$$fEh\succcurlyeq_1 gEh \iff \int_S u(f)dp^*\geq\int_S u(g)dp^* \iff fEh\succcurlyeq_2 gEh.$$

[13]Such triple x,y,z always exist. Let y be such that $u(y)=0$ and choose x,z so that $\nu_1(s)=\frac{u(z)}{u(x)}$.

Now take x, y, z with $u(x) > u(y)$ and $E \subseteq N$ such that

$$xEy \sim_1 z,$$

which is equivalent to $\gamma_1^*(E) = \frac{u(z)}{u(x)-u(y)}$. Since $E \subseteq N$, it follows that $\gamma_2^*(E) \geq \gamma_1^*(E)$, which is equivalent to

$$xEy \succsim_2 z,$$

as desired.

A.8 Proof of Theorem 5

First, we show that there is a $V : \mathcal{F} \to \mathbb{R}$ that represents $\succsim$. Note that axiom CI implies Risk Independence.[14] Thus, axioms WO, A and CI restricted to X allows to apply the Mixture Space theorem to obtain a non-constant and affine $u : X \to \mathbb{R}$ that represents $\succsim$ on X. By non-triviality there are $x^*, x_* \in X$ such that $x^* \succ x_*$ and, w.l.o.g., we can normalize $u(x^*) = 1$ and $u(x_*) = 0$.

Lemma 1. *For each $f \in \mathcal{F}$, there is $\gamma \in [0, 1]$ such that:*

1. $\gamma x^* + (1-\gamma)x_* \sim f$, *if* $x^* \succsim f \succsim x_*$,
2. $\gamma f + (1-\gamma)x_* \sim x^*$, *if* $f \succ x^*$,
3. $\gamma f + (1-\gamma)x^* \sim x_*$, *if* $x_* \succ f$.

Proof of Lemma 1. Take $f \in \mathcal{F}$, if $x^* \succsim f \succsim x_*$, by axiom A there is $\gamma \in [0, 1]$ such that $f \sim \gamma x^* + (1-\gamma)x_*$ (see the proof of Claim 1 in Cerreia-Vioglio et al., 2011). Suppose now that $f \succ x^*$ and define

$$U = \{\alpha \in (0, 1) : \alpha f + (1-\alpha)x_* \succ x^*\}$$

and

$$L = \{\beta \in (0, 1) : x^* \succ \beta f + (1-\beta)x_*\}.$$

By axiom A, U and L are non-empty. We show that $\alpha > \beta$ for all $\alpha \in U$ and $\beta \in L$. Suppose not, then for some $\alpha \in U, \beta \in L$, we have $\frac{\alpha}{\beta} \leq 1$. By CI, $\frac{\alpha}{\beta}x^* + (1-\frac{\alpha}{\beta})x_* \succ \frac{\alpha}{\beta}(\beta f + (1-\beta)x_*) + (1-\frac{\alpha}{\beta})x_* \sim \alpha f + (1-\alpha)x_* \succ x^*$. Since u is affine and represents $\succsim$ on X, it follows that $\frac{\alpha}{\beta} > 1$, a contradiction. The case $x_* \succ f$ can be handled similarly. □

Let define $\bar{\alpha} = \inf_{\alpha \in U} \alpha$ and $\bar{\beta} = \sup_{\beta \in L} \beta$. Since both U and L are non-empty, we have $1 > \bar{\alpha} \geq \bar{\beta} > 0$. There are three cases to check:

1. $\bar{\alpha} f + (1-\bar{\alpha})x_* \sim x^*$. In this case, the statement follows by setting $\gamma = \bar{\alpha}$.

[14] This axiom states that, if $x, y, z \in X$ and $\gamma \in (0, 1]$, $x \succsim y$ implies $\gamma x + (1-\gamma)z \succsim \gamma y + (1-\gamma)z$.

2. $\bar{\alpha} \in U$. Thus, $\bar{\alpha} f + (1-\bar{\alpha})x_* \succ x^*$. By axiom A, there is $\gamma \in (0,1)$ such that

$$\gamma(\bar{\alpha} f + (1-\bar{\alpha})x_*) + (1-\gamma)x_* \succ x^*$$

or, equivalently, $\gamma\bar{\alpha} f + (1-\gamma\bar{\alpha})x_* \succ x^*$. Therefore, $\gamma\bar{\alpha} \in U$. Since $\gamma < 1$, $\gamma\bar{\alpha} < \bar{\alpha}$ contradicting $\bar{\alpha} = \inf_{\alpha\in U} \alpha$.

3. $\bar{\alpha} \notin U$ and $\bar{\alpha} f + (1-\bar{\alpha})x_* \not\succ x^*$. By axiom WO, $x^* \succ \bar{\alpha} f + (1-\bar{\alpha})x_*$, so that $\bar{\alpha} \in L$. However, $\bar{\alpha} \geq \bar{\beta} = \sup_{\beta\in L} \beta \geq \bar{\alpha}$, thus $\bar{\alpha} = \bar{\beta}$. By axiom A, there is $\gamma \in (0,1)$ such that

$$x^* \succ \gamma f + (1-\gamma)(\bar{\beta} f + (1-\bar{\beta})x^*$$

or equivalently $x^* \succ (\gamma + (1-\gamma)\bar{\beta})f + (1-\gamma)(1-\bar{\beta})x_*$. Therefore, $\gamma + (1-\gamma)\bar{\beta} \in L$, but $\gamma + (1-\gamma)\bar{\beta} > \bar{\beta}$, contradicting $\bar{\beta} = \sup_{\beta\in L} \beta$.

By Lemma 1, we can define, for any $f \in \mathcal{F}$, $V(f) = \gamma u(x^*) + (1-\gamma)u(x_*) = \gamma$ if $x^* \succcurlyeq f \succcurlyeq x_*$, $V(f) = [u(x^*) - (1-\gamma)u(x_*)]/\gamma = 1/\gamma$ if $f \succ x^*$, and $V(f) = -[u(x_*) - (1-\gamma)u(x^*)]/\gamma = -(1-\gamma)/\gamma$ if $x_* \succ f$. Clearly, if $f = x$, $V(f) = u(x)$.

Now, we prove that $V(f)$ is independent of x^* when $f \succ x^*$. Suppose there is $y \in X$ with $x^* \succcurlyeq y \succcurlyeq x_*$ and $\gamma' \in [0,1]$ such that $\gamma' f + (1-\gamma')x_* \sim y$. Thus, we can define $V(f) = V(y)/\gamma'$. By CI, for any $\alpha \in [0,1]$, $\alpha(\gamma f + (1-\gamma)x_*) + (1-\alpha)x_* \sim \alpha x^* + (1-\alpha)x_*$. Now, take β such that $\beta x^* + (1-\beta)x_* \sim y$. Then, $\beta(\gamma f + (1-\gamma)x_*) + (1-\beta)x_* \sim y$. This implies $\beta\gamma = \gamma'$. Moreover, $V(y) = \beta V(x^*) + (1-\beta)V(x_*) = \beta$ and then $V(f) = V(y)/\gamma' = 1/\gamma$. The case $x_* \succ f$ is analogous.

Now, we show that V is constant-linear, i.e., for all $f \in \mathcal{F}$, $x \in X$ and all $\gamma \in [0,1]$, $V(\gamma f + (1-\gamma)x) = \gamma V(f) + (1-\gamma)V(x)$. Take $f \in \mathcal{F}$ and $x \in X$. Suppose that $x^* \succcurlyeq f, x \succcurlyeq x_*$ and call it Case 1. By Lemma 1, there is $x_f \in X$ such that $x_f \sim f$. Then, by CI, for all $\gamma \in [0,1]$, $\gamma f + (1-\gamma)x \sim \gamma x_f + (1-\gamma)x$. Thus, $V(\gamma f + (1-\gamma)x) = V(\gamma x_f + (1-\gamma)x) = u(\gamma x_f + (1-\gamma)x) = \gamma u(x_f) + (1-\gamma)u(x) = \gamma V(f) + (1-\gamma)V(x)$. Assume now that $f \succ x^* \succcurlyeq x \succcurlyeq x_*$ (Case 2). By Lemma 1, there is $\beta_f \in [0,1]$ such that $\beta_f f + (1-\beta_f)x_* \sim x^*$. Take an arbitrary $\gamma \in [0,1]$, then by CI,

$$\gamma(\beta_f f + (1-\beta_f)x_*) + (1-\gamma)(\beta_f x + (1-\beta_f)x_*) \sim \gamma x^* + (1-\gamma)(\beta_f x + (1-\beta_f)x_*)$$

Moreover, $x^* \succcurlyeq x \succcurlyeq x_*$ implies that there is $\gamma_x \in [0,1]$ such that $\gamma_x x^* + (1-\gamma_x)x_* \sim x$. Therefore, CI again implies

$$\gamma(\beta_f f + (1-\beta_f)x_*) + (1-\gamma)(\beta_f x + (1-\beta_f)x_*) \sim \gamma x^* + (1-\gamma)(\beta_f(\gamma_x x^* + (1-\gamma_x)x_*) + (1-\beta_f)x_*)$$

or

$$\beta_f(\gamma f + (1-\gamma)x) + [\gamma(1-\beta_f) + (1-\gamma)(1-\beta_f)]x_* \sim (\gamma + (1-\gamma)\beta_f\gamma_x)x^* + [(1-\gamma)(1-\gamma_x) + (1-\beta_f)]x_*.$$

By the definition of V,

$$V(\gamma f + (1-\gamma)x) = \frac{\gamma + (1-\gamma)\beta_f \gamma_x}{\beta_f},$$

and $V(f) = \frac{1}{\beta_f}$. Lastly, $V(x) = \gamma_x$, thus

$$\gamma V(f) + (1-\gamma)V(x) = \gamma\frac{1}{\beta_f} + (1-\gamma)\gamma_x = V(\gamma f + (1-\gamma)x).$$

The case $x^* \succcurlyeq x \succcurlyeq x^* \succ f$ is analogous.

Suppose that $f \succ x \succ x^*$. Take $\beta \in [0,1]$ such that $\beta f + (1-\beta)x_* \sim x^*$. For arbitrary $\gamma \in [0,1]$, by CI,

$$\gamma(\beta f + (1-\beta)x_*) + (1-\gamma)(\beta x + (1-\beta)x_*) \sim \gamma x^* + (1-\gamma)(\beta x + (1-\beta)x_*)$$

or

$$\beta(\gamma f + (1-\gamma)x) + (1-\beta)x_* \sim \gamma x^* + (1-\gamma)(\beta x + (1-\beta)x_*).$$

Since $f \succ x$, $x^* \succ \beta x + (1-\beta)x_*$, then there is $\gamma_x \in [0,1]$ such that $\beta x + (1-\beta)x_* \sim \gamma_x x^* + (1-\gamma_x)x_*$. Again by CI

$$\begin{aligned}\beta(\gamma f + (1-\gamma)x) + (1-\beta)x_* &\sim \gamma x^* + (1-\gamma)(\gamma_x x^* + (1-\gamma_x)x_*)\\ &= (\gamma + (1-\gamma)\gamma_x)x^* + (1-\gamma)(1-\gamma_x)x_*\end{aligned}$$

By definition of V, $V(\gamma f + (1-\gamma)x) = \frac{\gamma + (1-\gamma)\gamma_x}{\beta}$, and $V(f) = \frac{1}{\beta}$. Lastly, $V(\beta x + (1-\beta)x_*) = \gamma_x$ and thus $V(x) = \frac{\gamma_x}{\beta}$. It follows that

$$V(\gamma f + (1-\gamma)x) = \frac{\gamma + (1-\gamma)\gamma_x}{\beta} = \gamma V(x) + (1-\gamma)V(y) = \gamma\frac{1}{\beta} + (1-\gamma)\frac{\gamma_x}{\beta}.$$

An analogous argument applies to $x \succ f \succ x^*$ and to $x_* \succ x \succ f$ or $x_* \succ f \succ x$. Suppose that $f \succ x^* \succ x_* \succ x$. By Lemma 1, there is $\alpha \in [0,1]$ such that $\alpha f + (1-\alpha)x_* \sim x^*$ and $\beta \in [0,1]$ such that $\beta x + (1-\beta)x^* \sim x_*$. There are two cases: $\alpha \leq \beta$ and $\alpha > \beta$. Suppose that $\alpha \leq \beta$. Then, for any $\gamma \in (0,1]$,

$$\gamma(\alpha f + (1-\alpha)x_*) + (1-\gamma)(\alpha x + (1-\alpha)x^*) \sim \gamma x^* + (1-\gamma)(\alpha x + (1-\alpha)x^*). \tag{6}$$

Since $\alpha \leq \beta$, $\alpha x + (1-\alpha)x^* \succcurlyeq \beta x + (1-\beta)x^* \sim x_*$, and there is $z \in X$ such that $z \sim \alpha x + (1-\alpha)x^*$. Thus, there is $\gamma_z \in [0,1]$ such that $z \sim \gamma_z x^* + (1-\gamma_z)x_*$. By CI,

$$\gamma(\alpha f + (1-\alpha)x_*) + (1-\gamma)(\alpha x + (1-\alpha)x^*) \sim \gamma x^* + (1-\gamma)(\alpha x + (1-\alpha)x^*) \sim \gamma x^* + (1-\gamma)z.$$

The left-hand side of equation (6) is equal to $\alpha(\gamma f + (1-\gamma)x) + (1-\alpha)(\gamma x_* + (1-\gamma)x^*)$ and the right-hand side is indifferent to $(\gamma + (1-\gamma)\gamma_z)x^* + (1-\gamma)(1-\gamma_z)x_*$. Therefore, $V(\gamma f + (1-\gamma)x) = \frac{\gamma + (1-\gamma)\gamma_z - (1-\alpha)(1-\gamma)}{\alpha}$. Moreover, $V(f) = 1/\alpha$ and since $\alpha x + (1-\alpha)x^* \sim \gamma_z x^* + (1-\gamma_z)x_*$, $V(x) = (\gamma_z - 1 + \alpha)/\alpha$. Therefore, $\gamma V(f) + (1-\gamma)V(x) = \gamma\frac{1}{\alpha} + (1-\gamma)\frac{\gamma_z - 1 + \alpha}{\alpha} = V(\gamma f + (1-\gamma)x)$. The case $\alpha > \beta$ is similar. Lastly, an analogous argument applies to the remaining case $x \succ x^* \succ x_* \succ f$.

To prove part 1, consider $x, y \in X$ and $A \subseteq S$. Suppose that $x^* \succcurlyeq x \succ y \succcurlyeq x_*$. Then, there are $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ such that $x \sim \alpha x^* + (1-\alpha)y$ and $y \sim \beta x^* + (1-\beta)x_*$. $A \subseteq S$,

$$\begin{aligned}
V(xAy) =& V((\alpha x^* + (1-\alpha)y)Ay) = V(\alpha x^* Ay + (1-\alpha)y) \\
=& \alpha V(x^* Ay) + (1-\alpha)u(y) \\
=& \alpha V(x^* A(\beta x^* + (1-\beta)x_*)) + (1-\alpha)u(y) \\
=& \alpha V(\beta x^* + (1-\beta)(x^* Ax_*)) + (1-\alpha)u(y) \\
=& \alpha\beta u(x^*) + \alpha(1-\beta)V(x^* Ax_*) + (1-\alpha)u(y) \\
=& \alpha\beta + \alpha(1-\beta)V(x^* Ax_*) + (1-\alpha)u(y),
\end{aligned}$$

where the first and forth equalities follow from axiom IS (i.e., if $x \sim y$ then $xAz \sim yAz$ for all $A \subseteq S$ and $z \in X$) and the other equalities from the properties of V. Since $y \sim \beta x^* + (1-\beta)x_*$, $u(y) = \beta$, thus

$$V(xAy) = \alpha u(y) + \alpha(1-\beta)V(x^* Ax_*) + (1-\alpha)u(y)$$

Moreover, $x \sim \alpha x^* + (1-\alpha)y$, thus $u(x) = \alpha + (1-\alpha)\beta$ so that $u(x) - u(y) = \alpha(1-\beta)$, and

$$V(xAy) = u(y) + (u(x) - u(y))V(x^* Ax_*).$$

The remaining cases, (i) $x \succcurlyeq x^*, y \succcurlyeq x_*$, (ii) $x^* \succ x, x_* \succ y$, and (iii) $x \succ x^*, x_* \succ y$ can be handled similarly, so we omit the proofs. Lastly, we define $\rho(A) = V(x^* Ax_*)$. By IS, $x^* \sim x^* Sx_*$, thus $\rho(S) = V(x^* Sx_*) = u(x^*) = 1$. Similarly, $\rho(\emptyset) = V(x^* \emptyset x_*) = u(x_*) = 0$, thus ρ is a signed capacity.

Sufficiency of the axioms WO, A, IS and CI is straightforward.

A.9 Proof of Proposition 3

Suppose that $\succcurlyeq$ has a SIB representation. Let define the set function $\rho : 2^S \to \mathbb{R}$ as $\rho(E) = V(x^* Ex_*)$ where $x^*, x_* \in X$ are such that $V(x^*) = 1$ and $V(x_*) = 0$. By definition, $\rho(S) = 1$ and $\rho(\emptyset) = 1 - \rho(S) = 0$, thus ρ is a signed capacity. By Cerreia-Vioglio et al. (2012, Proposition 7), any signed capacity of bounded variation ρ can be decomposed into $\rho = \rho^+ - \rho^-$, where ρ^+ and ρ^- are capacities. Our signed capacity ρ is clearly of bounded variation since the state space is finite. Thus, we can write $\rho = a\mu^+ - b\mu^-$, where $a = \rho^+(S)$, $\mu^+ = \rho^+/a$, $b = \rho^-(S)$, and $\mu^- = \rho^-/b$. With these definitions, we get $\mu^+(S) = \mu^-(S) = 1$, and the normalization $\rho(S) = 1$ implies $a - b = 1$. To obtain the decomposition above, we then set $\Gamma(E) = b(\mu^+(E) - \mu^-(E))$ for any $E \subseteq S$. Thus, for all $x \succcurlyeq y$ in X and all $E \subseteq S$, $V(xEy) = \rho(E)u(x) + (1 - \rho(E))u(y)$ or, equivalently, $V(xEy) = (\mu^+(E) + \Gamma(E))u(x) + (1 - \mu^+(E) - \Gamma(E))u(y)$ yielding the generalized valence representation.

Suppose that $\succcurlyeq$ has a generalized valence representation. Then, by defining $\rho : 2^S \to \mathbb{R}$ as $\rho(E) = \mu(E) + \Gamma(E)$, we obtain a SB representation of $\succcurlyeq$. Indeed, $\rho(S) = \mu(S) + \Gamma(S) = 1$ and $\rho(\emptyset) = \mu(\emptyset) + \Gamma(\emptyset) = 0$.

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