Reevaluating the Shapley Value: A New Justification and Extension Online Appendix

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A. Uniqueness of the α-Procedure

Our α-procedure is the unique procedure within a general class of procedures that yield the Shapley value and satisfy our Priority Axiom and a Monotonicity Axiom.[1] The Monotonicity Axiom, formalized below, requires that players with larger marginal contributions receive at least as much as those with smaller marginal contributions.[2]

We now allow for a more general division of the $1-\alpha$ share of value that goes to the members of the coalition being joined. As before, when player i joins $S\backslash i$, its marginal contribution is partitioned into $j^*(i)+1$ intervals. When the marginal contributions are distinct, $j^*(i)+1=i$. Since a general weighting system has to allow for that possibility, we partition m_i into i intervals (and recognize that intervals past $j^*(i)+1$ will be of length 0).

Under the Interval Equality Axiom, each interval is divided equally among the players with a claim to that interval. Here we allow for unequal divisions. For each $k \le i$, we specify weights $w_j(S,i;k)$ that determine player j's share of the kth interval, $[m_{k-1}, m_k)$, when player i joins $S\backslash i$. If two or more players are tied, we order them arbitrarily, as before. To support the interpretation in terms of weights, we require for each $k \le i$:

$$\sum_{j \in S} w_j(S,i;k) = 1. \tag{A.1}$$

The Priority Axiom still applies. Thus, the players being joined receive all of $(1-\alpha)m_i$, except in the case $i=|S|$. This implies that $w_j(S,j;k)=0$ for all j,k except when $j=k=|S|$, in which case $w_j(S,j;k)=1$.

The Interval A.1uality Axiom specifies: $w_j(S,i;k)=\frac{1}{|S|-k}$ for $j \ne i$ and $k \le \min\{j,i\}$. For this set of $\{i,j,k\}$, the weights in each interval are constant across i and j. Here, the weights in each interval can vary with i and j, and a player can share in intervals beyond its claim. For example, weights can be proportional to the player's rank in the ordering. Formally, for $j \ne i$ and $k \le i$:

$$w_j(S,i;k) = \frac{j}{|S|(|S|+1)/2 - i}. \tag{A.2}$$

In this case, player j's weights are positive even for intervals $k > j$.

Alternatively, weights could be proportional to the player's rank, but the division is limited to the players who have a claim on a given interval. (This ensures a dummy player receives 0.) Formally, for

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[1] We thank Sergiu Hart for encouraging us to consider this question.

[2] The monotonicity is across players. The Young (1985) monotonicity axiom operates across games.

$j \neq i$ and $k \leq i$:

$$w_j(S,i;k) = \begin{cases} 0 & \text{if } j < k; \\ \frac{j}{(|S|+1-k)(|S|+k)/2-i} & \text{if } j \geq k. \end{cases} \tag{A.3}$$

Another possibility is that all the weight could go to the player in $S\backslash i$ with the highest rank. Formally, for $j \neq i$ and $k \leq i$:

$$w_j(S,i;k) = \begin{cases} 1 & \text{if } j = \max\{S\backslash i\}; \\ 0 & \text{otherwise.} \end{cases} \tag{A.4}$$

It can be checked that each of these examples satisfies Equation A.1 for $k \leq i$, as required. It is also true that in each of these examples, the weights are weakly increasing with a player's rank. Our Monotonicity Axiom requires that this holds in general.

Monotonicity Axiom: The weights $w_j(S,i;k)$ are weakly increasing in $j \in S\backslash i$.

Intuitively, if player j's marginal contribution to S is larger than that of player j', then player j is more powerful than player j' and should therefore receive more value in each interval $[m_{k-1}, m_k)$.

We denote the weights in our α-procedure by $w_j^*(S,i;k)$:

$$w_j^*(S,i;k) = \begin{cases} \dfrac{1}{|S|-k} & \text{if } j \neq i, k \leq \min\{i,j\}; \\ 1 & \text{if } i = j = k = |S|; \\ 0 & \text{otherwise.} \end{cases} \tag{A.5}$$

These weights satisfy the Interval Equality and Priority Axioms, and by Theorem 1 and Corollary 1.1, they yield the Shapley value as payoffs. Interval Equality is stronger than Monotonicity and therefore implies the Monotonicity axiom is satisfied. While other weighting systems satisfy the Priority and Monotonicity Axioms, only one such weighting scheme always leads to the Shapley value.

Theorem A1 *Under the Priority and Monotonicity Axioms, the weights $w_j^*(S,i;k)$ of the α-procedure uniquely yield the Shapley value.*

Proof. The proof of is based on a repeated application of Unanimity Games, one for each interval k. The details are straightforward, but lengthy. To begin, recall from Theorem 1 and Corollary 1.1 that the α-procedure leads to the Shapley value. Also, these weights satisfy the Priority and Monotonicity Axioms. Therefore, we need to prove that no other weighting system leads to the Shapley value. For any fixed α, the expected payoffs to the players are an $\alpha : 1-\alpha$ weighted average of the Shapley value and the $\alpha = 0$ solution. Thus, we only need to establish necessity of the weights in Equation A.5 for the case $\alpha = 0$.

We first suppose that $S = N$. The proof proceeds in two steps. In Step 1, we find conditions on the weights so that the expected payoffs are the Shapley values. In Step 2, we show that imposing the Priority and Monotonicity Axioms reduces these weights to those given by Equation A.5.

Step 1: Consider the Unanimity Game, i.e., the game where $v(N) > 0$ and $v(S) = 0$ for $S \subsetneq N$. Note that the characteristic function is 0 everywhere in any subgame. The Shapley value is $v(N)/|N|$ for all players. In this game, there is only one non-zero interval of value, namely, $[0, m_1(N)] = [0, v(N)]$. The expected payoff to player j is therefore given by (recalling $\alpha = 0$):

$$\pi^j(N;0) = \frac{1}{|N|}\sum_{i\in N}\pi_j(N,i;0) = \frac{1}{|N|}\sum_{i\in N}\sum_{k=1}^{1} w_j(N,i;k)(m_k - m_{k-1}) = \frac{m_1}{|N|}\sum_{i\in N} w_j(N,i;1). \tag{A.6}$$

The payoffs in all subgames are 0. Therefore, if player j is to receive its Shapley value, it is necessary that:

$$\sum_{i\in N} w_j(N,i;1) = 1 \text{ for all } j \in N. \tag{A.7}$$

Also, since weights sum to 1, we must also gave:

$$\sum_{j \in N} w_j(N, i; 1) = 1 \text{ for all } i \in N. \tag{A.8}$$

We refer to Equation A.7 as the **Shapley condition** and to Equation A.8 as the **Sum condition**.

Step 2: We now make use of the Priority and Monotonicity Axioms. The case $|N| = 1$ is trivial. Both the Shapley and Sum conditions imply $w_1(N, 1; 1) = 1$, which is exactly Equation A.5 when $|N| = 1$. So, we assume $|N| \geq 2$. Priority implies $w_j(N, j; 1) = 0$, so that the Shapley condition becomes:

$$\sum_{i \in N \setminus j} w_j(N, i; 1) = 1 \text{ for all } j \in N, \tag{A.9}$$

and the Sum condition becomes:

$$\sum_{j \in N \setminus i} w_j(N, i; 1) = 1 \text{ for all } j \in N. \tag{A.10}$$

We now alternate between applying the Sum condition and the Shapley condition. Monotonicity and the Sum condition imply:

$$(|N| - 1)w_1(N, i; 1) \leq \sum_{j \in N \setminus i} w_j(N, i; 1) = 1 \text{ for } i \neq 1, \tag{A.11}$$

from which:

$$w_1(N, i; 1) \leq \frac{1}{|N| - 1} \text{ for } i \neq 1. \tag{A.12}$$

If $w_1(N, i; 1) < 1/(|N| - 1)$ for any $i \neq 1$, then:

$$\sum_{i \in N \setminus 1} w_1(N, i; 1) < 1, \tag{A.13}$$

which violates the Shapley condition A.9. Therefore, we must have:

$$w_1(N, i; 1) = \frac{1}{|N| - 1} \text{ for } i \neq 1. \tag{A.14}$$

By Monotonicity and Equation A.14:

$$w_j(N, i; 1) \geq w_1(N, i; 1) = 1/(|N| - 1) \text{ for } i \notin \{1, j\}. \tag{A.15}$$

Using the Sum condition A.10, we conclude that:

$$w_j(N, i; 1) = \frac{1}{|N| - 1} \text{ for all } j \in N \text{ and } i \notin \{1, j\}. \tag{A.16}$$

The preceding argument covers all weights in the first interval of value, excepting the weights $w_j(N, 1; 1)$ for $j \geq 2$. Returning to the Shapley condition A.9, we can write:

$$\sum_{i \in N \setminus j} w_j(N, i; 1) = w_j(N, 1; 1) + \frac{|N| - 2}{|N| - 1} = 1 \text{ for } j \geq 2, \tag{A.17}$$

which implies:

$$w_j(N, 1; 1) = \frac{1}{|N| - 1} \text{ for } j \geq 2. \tag{A.18}$$

We have now established that, under Priority and Monotonicity, weights that lead to the Shapley value must satisfy:

$$w_j(N,i;k) = \begin{cases} \dfrac{1}{|N|-k} & \text{if } j \neq i, k = 1; \\ 1 & \text{if } i = j = k = |N|, k = 1. \end{cases} \tag{A.19}$$

Equation A.19 coincides with Equation A.5 when $k = 1$. It remains to determine necessary conditions on weights for $k \geq 2$. To do so, consider a game in which player 1's only (strictly) positive marginal contribution is $m_1(N)$; its marginal contribution to all other coalitions is 0. Thus, player 1's Shapley value for this game is $m_1/|N|$. Suppose that the marginal contributions $m_j(N)$ for all other players $j \neq 1$ are all larger than $m_1(N)$ and are all distinct. Assume player 1's payoff is the Shapley value when $\alpha = 0$.

We established above that player 1's payoff from the first interval of value is $m_1/|N|$. It follows that the weights for player 1 from all higher intervals must be 0, i.e., $w_1(N,i;k) = 0$ for $k \geq 2$. This completes the argument that the weights $w_1(N,i;k)$, for all i and k, are equal to those in Equation A.5. This is for player $j = 1$.

We now repeat this exercise by considering games in which player 1 is a dummy player. Analogously to above, we consider the Unanimity Game (strictly, subgame) among players $2,\ldots,N$. If $|N| = 2$, Priority implies $w_2(N,2,2) = 1$. For $N \geq 3$, we repeat our earlier argument to obtain the analog to Equation A.18:

$$w_j(N,i,2) = \frac{1}{|N|-2} \text{ for } j \neq i, j \geq 2. \tag{A.20}$$

As before, we next consider a game in which player 1 is a dummy player, player 2's only positive marginal contribution is $m_2/|N|$, and the marginal contributions for all other players are all larger than $m_2(N)$ and are all distinct. Player 2's Shapley value for this game is $m_2/|N|$. Assume player 2's payoff is also the Shapley value when $\alpha = 0$. Then, Equation A.20 implies that player 2's payoff from the second interval is $m_2/|N|$. It follows that the weights for player 2 from all higher intervals must be 0, i.e., $w_2(N,i;k) = 0$ for $k \geq 3$. This completes the argument that the weights $w_2(N,i;k)$, for all i and k, are equal to those in Equation A.5. This is for player $j = 2$.

We next move to player 3, by making both players 1 and 2 dummy players. In this manner, we establish that the weights $w_j(N,i;k)$, for all i, k, and j, are equal to those in Equation A.5.

Continuation: We can re-run the preceding argument for any $S \subsetneq N$, thereby completing the proof of Theorem 3. □

Absent our Monotonicity Axiom, there are other weights that lead to the Shapley value, as the following example shows.

Example A.1. *Let $S = \{1,2,3\}$ and fix the weights $w_j(S,i;1)$ for the first interval to be:*

$$w_1(S,2;1) = 1, \;\; w_3(S,2;1) = 0, \tag{A.21}$$

$$w_1(S,3;1) = 0, \;\; w_2(S,3;1) = 1, \tag{A.22}$$

$$w_2(S,1;1) = 0, \;\; w_3(S,1;1) = 1, \tag{A.23}$$

with all other weights given by Equation A.5. Focus on the first interval. When player 2 joins $\{1,3\}$, player 1 receives $(1-\alpha)m_1$ and player 3 receives 0. When player 3 joins $\{1,2\}$, player 1 receives 0 and player 2 receives $(1-\alpha)m_1$. When player 1 joins $\{2,3\}$, player 2 receives 0 and player 3 receives $(1-\alpha)m_1$. Summing across the three cases, each player receives a total of $(1-\alpha)m_1$, which is the same as under the α-procedure. All other payoffs are also the same as under the α-procedure, since the weights are the same. The result is that each player receives their Shapley value.

In our view, the counterintuitive aspect of this example is that when player 2 joins $\{1,3\}$, the weaker player 1 receives the full amount $(1-\alpha)m_1$, while the stronger player 3 receives 0. Of course, this shows that the weights in Example A.1 do not satisfy our Monotonicity Axiom.

B. Extension to NTU Games

We show how our α-procedure introduced in Brandenburger and Nalebuff (2025) can be extended to analyze NTU games. For $\alpha = 1$, this problem has been addressed in Hart and Mas-Colell (1996), who provide a procedure that yields the consistent Shapley value Maschler and Owen (1989, 1992). When $0 \le \alpha < 1$, a new procedure is required, just as in the TU case, since we need a rule for allocating the $1 - \alpha$ fraction of value. We achieve this by defining the NTU marginal contribution of a player to a coalition S, which leads to our generalized procedure.

Given an NTU game (N, V), we assume that the feasible sets $V(S)$ satisfy the standard conditions on the characteristic function; see, in particular, conditions (A.1)-(A.3) in Hart and Mas-Colell (1996). Let $\partial V(S)$ denote the boundary of the feasible set for S. For convenience, we perform two normalizations. We set $\partial V(\{i\}) = 0$ for all i. We also scale the utilities for all players so that the maximum feasible utility level of each player i in $V(N)$ is 1.

We begin with the case where $\partial V(N)$ is a hyperplane (therefore, the unit simplex under our scaling) and then show how to extend our analysis to the general convex case as in Maschler and Owen (1989, 1992). Let Ψ denote the tuple of payoffs from the procedure. Our normalization implies $\Psi(i) = 0$ for one-player games.

Assume inductively that we have a solution for coalitions of size up to $|N| - 1$ (for any characteristic function). Fix a game (N, V). We derive the solution for the set N. The marginal contributions associated with N are defined by:

$$d^i(N) = \max\{c^i : (c^i, \Psi(N\backslash i)) \in V(N)\}, \tag{B.1}$$

This quantity is the maximum possible payoff to player i given that the other players obtain their payoffs in the game without i. Our definition of marginal contribution differs from that in Hart and Mas-Colell (1996) in two ways. First, the marginal contributions as defined in Equation B.1 are independent of the order in which players arrive. In Hart and Mas-Colell (1996), the marginal contributions are defined inductively based on a specific ordering of player arrivals. Second, outside of a hyperplane game—that, is, a game where $V(S)$, for each $S \subseteq N$, is a half space—the solution to the subgame $\Psi(N\backslash i)$ need not be the average marginal contribution of each player in that game.

With our set of marginal contributions, and following our earlier convention, we index the players in order of increasing marginal contribution. The inductive step in our NTU procedure is obtained by adapting our earlier TU game.

From the set N we randomly select a player to be at risk. Given player i is at risk, we assign the probability τ_{ji} that player j is the proposer as in Equation (6) in Brandenburger and Nalebuff (2025), substituting the marginal contribution $d^i(S)$ for each $m_i(S)$, and then dividing by $d^i(S)$:[3]

$$\tau_{ji}(S) = \begin{cases} \frac{(1-\alpha)}{d^i} \displaystyle\sum_{k=1}^{\min\{i,j\}} \frac{d^k - d^{k-1}}{|S| - k} & \text{if } j \neq i; \\ \alpha & \text{if } j = i < |S|; \\ \alpha + \frac{(1-\alpha)}{d^{|S|}}(d^{|S|} - d^{|S|-1}) & \text{if } j = i = |S|. \end{cases} \tag{B.2}$$

In this way, the bargaining parameter α enters our NTU-procedure. The procedure, contingent on the random selection of i and j, assigns to everyone their value in $\Psi(N\backslash i)$, where the proposer i receives an additional amount $d^i(N)$. (As before, we extend $\Psi(N\backslash i)$ so that player i, who is not a member of $N\backslash i$, receives 0 under $\Psi(N\backslash i)$.) Because $\partial V(N)$ is a hyperplane (normalized to the unit simplex), it is always efficient and feasible to assign $d^i(N)$ to the player making the proposal. The payoffs $\Psi(N)$ are the expected values, where each player has an equal chance of being at risk:

$$\Psi(N) = \frac{1}{|N|} \sum_{i=1}^{N} \left[\Psi(N\backslash i) + \sum_{j=1}^{N} \tau_{ji} d^i(N\backslash i) e^j\right], \tag{B.3}$$

where e^j is the jth unit vector. Again, since $\partial V(N)$ is a hyperplane, this expected value is both efficient and feasible. The same argument as in the proof of Theorem 1 in Brandenburger and Nalebuff (2025)

[3] If some of the $d^i(S)$ are equal, we may be adding some extra intervals of length 0.

establishes:

$$\frac{1}{|N|}\sum_{i=1}^{N}\tau_{ji}d^i(N\backslash i) = d^j(N\backslash i), \tag{B.4}$$

from which:

$$\Psi(N) = d(N) + \frac{1}{|N|}\sum_{i=1}^{N}\Psi(N\backslash i), \tag{B.5}$$

where $d(N)$ is the tuple made up of the $d^j(N)$'s. We can see that this procedural formula is the NTU analog to the Shapley recursion formula.

This is the solution for the case where $\partial V(N)$ is a hyperplane. To find the solution for general $V(N)$, we look for a fixed point as in Maschler and Owen (1992). Start with a point p in the unit simplex. Consider the ray from the origin through p. This ray will intersect $\partial V(N)$ at some point q. Let hyperplane $H(q)$ be tangent to $\partial V(N)$ at q. Normalize $H(q)$ so that it is the unit simplex, and apply the same scaling to $V(N)$.

Now consider the game where the scaled $V(N)$ is extended to $H(q)$. Here, the boundary is a hyperplane, so we can apply the solution for $\Psi(N)$ from Equation B.5. This defines a continuous mapping from the unit simplex to itself, namely, from p to q to $\Psi(N)$. This mapping therefore has a fixed point. Moreover, the fixed point is a tangency point and thus lies on the boundary of (the scaled) $V(N)$. At the fixed point, the solution for $\Psi(N)$ is then defined as the consistent solution to the feasible set $V(N)$.

The intuition for selecting the fixed point is similar to that for the Independence of Irrelevant Alternatives Axiom of decision theory: $\Psi(N)$ is a solution for a larger set that includes $V(N)$ and it remains feasible in the smaller set $V(N)$, so we require it to be the solution for the smaller set.

Observe that the inductive step has two parts. We start with $|N|$ players and randomly break the set into $|N|-1$ inside players and one at-risk player. We apply the procedure to a game with $|N|-1$ players, and we divide up the at-risk player's marginal contribution to obtain the solution to a game with $|N|$ players. This first step is carried out when the boundary for $V(N)$ is a hyperplane. We then use the solution to all such games to find a fixed point for general $V(N)$. This approach is similar to the way the Nash bargaining solution (Nash, 1950) is constructed.

We offer some remarks on our NTU procedure. First, if the game is TU, the procedure leads to the same outcome as under our α-procedure in Brandenburger and Nalebuff (2025). Next, for two-person games, our NTU procedure leads to the Nash (1950) bargaining solution for all values of α. Indeed, when the boundary of the bargaining set is a line, the NTU procedure selects the midpoint $\Psi = 1/2[(\alpha, 1-\alpha) + (1-\alpha, \alpha)] = (1/2, 1/2)$. As in the Nash bargaining solution, the NTU procedure for convex sets selects the boundary point that is the midpoint of the tangent line at that boundary point. For $\alpha = 1$, our procedure leads to the same consistent solution(s) as in Hart and Mas-Colell (1996).[4] Any consistent solution is based on the solution to a hyperplane game, and our procedures coincide in hyperplane games when $\alpha = 1$.

[4] Hart and Mas-Colell allow for a penalty in the case of disagreement. Our solutions coincide when the penalty in Hart and Mas-Colell is 0.

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