Axiomatization of Signed Rényi Entropy with Two Applications*

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Abstract

We modify the Rényi (1961) axioms for entropy to apply to negative ("signed") measures as arise, for example, in phase-space representations of quantum mechanics. We obtain two new measures of (lack of) information about a system – which we propose as signed analogs to classical Shannon entropy and classical Rényi entropy, respectively. In our first application, we show that signed Rényi entropy witnesses non-classicality of a system. Specifically, a measure has at least one negative component if and only if there is an associated signed Rényi entropy which is negative. Signed Shannon entropy does not witness non-classicality this way. In our second application, we identify a parametric family of signed Rényi entropies that have the property of being weakly increasing under doubly stochastic transformations of signed measures.

1 Introduction

Rényi (1961) introduced a new definition of entropy generalizing Shannon entropy (Shannon, 1948), which he later used in various problems in probability theory and information theory. (See Rényi, 1970, pp.598-603 for applications to ergodicity of Markov chains and the central limit theorem.) Since then, Rényi entropy has been used in many fields (Csiszár, 2008). Two such fields are quantum foundations and quantum information, which are the primary motivation for the current paper. (An important application in quantum information is to uncertainty relations; see Coles et al., 2017 for a survey.)

The usual definition of entropy in quantum mechanics is von Neumann entropy (von Neumann, 1932), which is the natural analog to Shannon entropy. One can likewise define a quantum analog to Rényi entropy. These definitions are for the standard representation of quantum mechanics. The question in this paper is what are the natural definitions of Shannon and Rényi entropy in phase-space representations of quantum mechanics (Wigner, 1932; Feynman, 1987; Wootters, 1987; Gibbons, Hoffman, and Wootters, 2004). The issue is the negativity of phase-space probabilities – what are often called quasi-probabilities. Since both Shannon and Rényi entropies involve log terms, they can both become complex-valued in this case, which does not admit of an obvious interpretation in terms of the amount of uncertainty about a system.

In this paper, we return to the axiomatic basis of Rényi entropy (Rényi, 1961 and Daróczy, 1963) and modify the axioms so that entropy can retain its natural meaning in the presence of negative probabilities. In fact, we axiomatize what we will call signed Shannon and signed Rényi entropies for all signed measures, not just those normalized to sum to 1, to increase the scope of application of our theory. Interestingly, the usual relationship between Shannon entropy and Rényi entropy – namely, that Shannon entropy is the limit of Rényi entropy as the free parameter α in Rényi entropy tends to 1 – no longer holds in our case. That is, we obtain two quite distinct notions of entropy for signed measures.

A closely related paper to this one is Koukouledikis and Jennings (2022). These authors do not take an axiomatic approach, but, instead, directly restrict the α parameter of Rényi entropy so that the argument of the log in the Rényi formula is always a positive real number. Our entropy formula is defined for all $\alpha > 0$, but it coincides with the Koukouledikis-Jennings one when α is restricted as in their paper. (Also, we treat unnormalized measures.) Koukouledikis and Jennings (2022) show that negativity of Rényi entropy witnesses the presence of negative probabilities. That is, their entropy is negative if and only if there is at least one strictly negative probability present. Inspired by this result, we examine the same question in our first application of our axiomatization. We ask: Which of our signed Shannon and Rényi entropies witness non-classicality, i.e., negativity, of measures? The answer is:

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Our axiomatized signed Rényi entropy witnesses non-classical measures, but our axiomatized signed Shannon entropy does not do so (at least, not in the same way).

We see this property of signed Rényi entropy, as we define it by modifying the original Rényi (1961) axioms, as an important argument for its use in analysis of quantum systems – and in other applications where signed measures arise.

Our second application is to doubly stochastic transformations of signed measures. (Note the special case that a doubly stochastic transformation of a signed probability measure returns a signed probability measure.) It is known that ordinary Rényi entropy (for any $\alpha > 0$) is weakly increasing under doubly stochastic transformations of (non-negative) probability measures. (See Theorems 4 and 5 in Kvålseth, 2022, and also Baez, 2012.) We establish the following signed analog:

Our axiomatized signed Rényi entropy is weakly increasing under doubly stochastic transformations of signed measures, whenever the free parameter α is an even positive integer.

The condition that the parameter α is an even positive integer can be derived from a smoothness condition on signed Rényi entropy (see Brandenburger, La Mura, and Zoble, 2022). Doubly stochastic matrices are, of course, used in many fields, including quantum foundations and quantum information (Rovelli, 1996; Louck, 1997; Dunki and Życzkowski, 2009). We hope that the fact that the behavior of our signed Rényi entropy parallels the classical case – i.e., it is increasing under doubly-stochastic dynamics – means that it will find applications in phase-space representations of quantum systems and, perhaps, elsewhere.

Elsewhere (Brandenburger, La Mura, and Zoble, 2022), we employ signed Rényi entropy, where α is an even positive integer, to axiomatize the simplest quantum system, namely, the qubit, via an entropic uncertainty principle. Bruckner and Zeilinger (2009) present axioms that single out Tsallis entropy (Tsallis, 1988) of degree 2 as the unique measure of uncertainty of a quantum system. (Tsallis entropy is a monotonic transformation of Rényi entropy.) Rényi entropy with $\alpha = 2$, often called collision entropy, has been used in a series of recent papers by Aw et al. (2023), Onggadinata, Kurzynski, and Kaszlikowski (2023a,b), and Onggadinata et al. (2024) to analyze several questions posed about quantum or superquantum systems involving negative probabilities.

2 Axioms for Rényi Entropy

The approach followed by Rényi (1961) was first to axiomatize entropy for the class of non-negative measures with total weight less than or equal to 1 and then to specialize to probability measures. Fix a finite set of **states** $X = \{x_1, \ldots, x_n\}$. A **generalized probability measure** is Q on X is a tuple $Q = (q_1, \ldots, q_n)$ where each $q_i \in \mathbb{R}_+$ and $0 < w(Q) \stackrel{\text{def}}{=} \sum_{i=1}^n q_i \leq 1$. The quantity $w(Q)$ will be called the **weight** of Q.

Given two generalized probability measures $P = (p_1, \ldots, p_m)$ and $Q = (q_1, \ldots, q_n)$, we denote by $P * Q$ the generalized probability measure which is the direct product:

$$(p_1 q_1, \ldots, p_1 q_n, \ldots, p_m q_1, \ldots, p_m q_n), \tag{1}$$

whenever it is well-defined, i.e., whenever $\sum_{i,j} p_i q_j > 0$. Also, we denote by $P \cup Q$ the generalized probability measure which is the (direct) sum:

$$(p_1, \ldots, p_m, q_1, \ldots, q_n), \tag{2}$$

whenever it is well-defined, i.e., whenever $\sum_i p_i + \sum_j q_j \leq 1$. Finally, we write (q) for the generalized probability measure consisting of the single real number $q > 0$.

Rényi (1961) proposed the following axioms for entropy:

Axiom 1. ***(Symmetry)*** *$H(Q)$ is a symmetric function of the elements of Q.*

Axiom 2. ***(Continuity)*** *$H((q))$ is continuous for all $0 < q \leq 1$.*

Axiom 3. ***(Calibration)*** *$H((\frac{1}{2})) = 1$.*

Axiom 4. ***(Additivity)*** *$H(P * Q) = H(P) + H(Q)$ whenever $H(P * Q)$ is well-defined.*

Axiom 5. ***(Mean-Value Property)*** *There is a strictly monotone and continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for any P, Q, whenever $H(P \cup Q)$ is well-defined:*

$$H(P \cup Q) = g^{-1}\left[\frac{w(P)g(H(P)) + w(Q)g(H(Q))}{w(P \cup Q)}\right]. \tag{3}$$

Rényi (1961) proved that if the function g in Axiom 5 takes the form $g(x) = -dx + e$, for constants $d \neq 0$ and e, then Axioms 1-5 characterize the entropy functional:

$$H_1(W) \stackrel{\text{def}}{=} -\frac{\sum_{i=1}^n q_i \log_2 q_i}{\sum_{i=1}^n q_i}, \tag{4}$$

while, if $g(x) = -d2^{(1-\alpha)x} + e$ for $\alpha > 0$ with $\alpha \neq 1$, then Axioms 1-5 characterize the family of entropy functionals:

$$H_\alpha(Q) \stackrel{\text{def}}{=} -\frac{1}{\alpha - 1} \log_2 \Big[\frac{\sum_{i=1}^n q_i^\alpha}{\sum_{i=1}^n q_i}\Big]. \tag{5}$$

The entropies in Equations 4 and 5 are the familiar Shannon and Rényi entropies, defined for generalized probability measures. Rényi (1961) notes that an application of L'Hôpital's rule gives:

$$\lim_{\alpha \to 1} H_\alpha(Q) = H_1(Q). \tag{6}$$

Daróczy (1963) proved that Axioms 1-5 are, in fact, fully characterized by the two functional forms:

$$g(x) = -dx + e \text{ and } g(x) = -d2^{(1-\alpha)x} + e, \tag{7}$$

with the preceding parameter restrictions on d, e, and α. (He also observed that Axiom 1 is not needed, because it follows from Axiom 5.)

3 Extension to Signed Measures

In this section, we modify the Rényi axioms for the case of signed measures. A **signed measure** Q on X is a tuple $Q = (q_1, \ldots, q_n)$ where each $q_i \in \mathbb{R}$. Thus, the components q_i are no longer required to be non-negative. We impose $w(Q) \neq 0$ but not $w(Q) = 1$ (except when Q is a signed probability measure). The notations $P * Q$ and $P \cup Q$ have the same meanings as in the previous section and are well-defined when $\sum_{i,j} p_i q_j \neq 0$ and $\sum_i p_i + \sum_j q_j \neq 0$, respectively. We now state our set of axioms.

Axiom 0. ***(Real-Valuedness)*** *$H(Q)$ is a non-constant real-valued function of Q.*

Axiom 2′. ***(Continuity)*** *$H((q))$ is continuous for all $q \neq 0$.*

Axiom 3. ***(Calibration)*** $H((\frac{1}{2})) = 1$.

Axiom 4. ***(Additivity)*** *$H(P * Q) = H(P) + H(Q)$ whenever $H(P * Q)$ is well-defined.*

Axiom 5. ***(Mean-Value Property)*** *There is a strictly monotone and continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for any P, Q, whenever $H(P \cup Q)$ is well-defined:*

$$H(P \cup Q) = g^{-1}\Big[\frac{w(P)g(H(P)) + w(Q)g(H(Q))}{w(P \cup Q)}\Big]. \tag{8}$$

Some comments on the axioms. Axiom 0 can be viewed as the requirement that if entropy is to again to be interpreted as a measure of the amount of uncertainty or (lack of information) about a system, then it must be a real quantity. This axiom has bite when applied to signed vs. unsigned measures because simply allowing negative values in the usual Shannon or Rényi entropy formulas will yield complex values when we have to take the log of a negative number. Axiom 2′ is the natural analog to Axiom 2 for the signed case. (Recall that we do not allow $Q = (q)$ when $q = 0$.) In fact, this axiom can be relaxed, as we note in the proof of our characterization theorem to come. The remaining axioms are unchanged.

Theorem 1. *Axioms 0, 2′, 3, 4, and 5 hold if and only if $H(Q)$ is given by:*

$$H_{Sh}(Q : k) \stackrel{\text{def}}{=} -\frac{\sum_{i=1}^n |q_i| \log_2 |q_i|}{|\sum_{i=1}^n q_i|} - k\Big(\frac{\sum_{i=1}^n |q_i|}{|\sum_{i=1}^n q_i|} - 1\Big), \tag{9}$$

or:

$$H_\alpha(Q; k) \stackrel{\text{def}}{=} -\frac{1}{\alpha - 1} \log_2 \Big[\frac{\sum_{i=1}^n |q_i|^\alpha}{|\sum_{i=1}^n q_i|} + k\Big(\frac{\sum_{i=1}^n |q_i|}{|\sum_{i=1}^n q_i|} - 1\Big)\Big], \tag{10}$$

where k and α are free parameters with $\alpha > 0$ and $\alpha \neq 1$.

The proof of Theorem 1 is in Section 6. We make some observations. First, it is immediate that Equations 9 and 10 reduce to Equations 4 and 5, respectively, when all q_i's are non-negative. Second, in the signed case, it is not possible to obtain $H_{Sh}(Q;k)$, the analog to Shannon entropy, as the $\alpha \to 1$ limit of Rényi α-entropies. To see this, note that $H_\alpha(Q;k)$ will diverge in general unless:

$$\frac{\sum_{i=1}^n |q_i|}{|\sum_{i=1}^n q_i|} + k\big(\frac{\sum_{i=1}^n |q_i|}{|\sum_{i=1}^n q_i|} - 1\big) = 1, \tag{11}$$

We therefore try setting $k = -1$ so that Equation 11 is satisfied. Another application of L'Hôpital's rule then gives:

$$\lim_{\alpha\to 1} H_\alpha(Q;-1) = -\frac{\lim_{\alpha\to 1} \frac{\partial}{\partial\alpha} \log_2 \left[(\sum_{i=1}^n |q_i|^\alpha - \sum_{i=1}^n |q_i| + |\sum_{i=1}^n q_i|)/|\sum_{i=1}^n q_i|\right]}{\lim_{\alpha\to 1} \frac{\partial}{\partial\alpha}(\alpha - 1)} \tag{12}$$

$$= -\frac{1}{\log_e 2} \times \lim_{\alpha\to 1} \frac{\sum_{i=1}^n |q_i|^\alpha \log_e |q_i|}{\sum_{i=1}^n |q_i|^\alpha - \sum_{i=1}^n |q_i| + |\sum_{i=1}^n q_i|} \tag{13}$$

$$= -\frac{\sum_{i=1}^n |q_i| \log_2 |q_i|}{|\sum_{i=1}^n q_i|}. \tag{14}$$

Equation 14 will generally differ from Equation 9 for $k = -1$. In fact, we have shown:

$$\lim_{\alpha\to 1} H_\alpha(Q;-1) = H_{Sh}(Q;0). \tag{15}$$

We see that there is no consistent choice of the parameter k that yields Shannon entropy as a limiting case of Rényi entropy in the signed case. In what follows, we will set $k = 0$ which, as we will see, delivers a sharp characterization of non-classicality of measures. In making this choice, we accept that Rényi α-entropy diverges as $\alpha \to 1$. Rényi entropy and Shannon entropy are no longer nested (in the sense of a limit) once signed measures are admitted.

Definition 1. *Given a signed measure Q with $w(Q) \neq 0$, the* ***signed Shannon entropy*** *of Q is defined by:*

$$H_{Sh}(Q) = -\frac{\sum_{i=1}^n |q_i| \log_2 |q_i|}{|\sum_{i=1}^n q_i|}. \tag{16}$$

Definition 2. *Given a signed measure Q with $w(Q) \neq 0$, the* ***signed Rényi entropy*** *of Q is defined, for $\alpha > 0$ with $\alpha \neq 1$, by:*

$$H_\alpha(Q) = -\frac{1}{\alpha - 1} \log_2 \big[\frac{\sum_{i=1}^n |q_i|^\alpha}{|\sum_{i=1}^n q_i|}\big]. \tag{17}$$

In their extension of Rényi entropy to signed probabilities, Koukouledikis and Jennings (2022) assume that $\alpha = 2a/(2b-1)$, where a, b are positive integers with $a \geq b$, so that q_i^α is non-negative real-valued and the usual Rényi formula remains real-valued. Of course, in this same case, we can drop the absolute-value operations in Equation 17. Moreover, if Q is a signed probability measure, then, again in this case, Equation 17 further reduces to the usual definition of Rényi entropy, and our definition coincides with the Koukouledikis-Jennings one.

4 Witnessing Signed Measures

In this section, we show that signed Rényi entropy as just defined witnesses non-classicality of probability measures or of measures more generally. This result is analogous to and also extends the domain of Theorem 10 in Koukouledikis and Jennings (2022). As we will see, signed Shannon entropy does not witness non-classicality – at least, not in this way.

Theorem 2. *A (signed) measure $Q = (q_1, \ldots, q_n)$ contains at least one strictly negative component q_i if and only if there is a value of the free parameter α such that the Rényi α-entropy $H_\alpha(Q)$ is strictly negative.*

Proof. Suppose $q_i < 0$ for some $i = 1, \ldots, n$. Then $\sum_{i=1}^n |q_i| > |\sum_{i=1}^n q_i|$. It follows that we can find positive numbers ϵ and r such that:

$$\frac{\sum_{i=1}^n |q_i|^{1+\epsilon}}{|\sum_{i=1}^n q_i|} = r > 1, \tag{18}$$

But then $H_{1+\epsilon}(Q) = -1/\epsilon \times \log_2(r)$, which is less than 0.

For the converse, consider a general (unnormalized) non-negative measure Q. We can copy the usual argument made for non-negativity of Rényi entropy in the case of probability measures. If $0 < \alpha < 1$, then $\sum_{i=1}^{n} q_i^\alpha \geq \sum_{i=1}^{n} q_i$, and, if $\alpha > 1$, then $\sum_{i=1}^{n} q_i^\alpha \leq \sum_{i=1}^{n} q_i$, so that in both cases $H_\alpha(Q) \geq 0$. (In this direction, we could also treat the case $\alpha = 1$, although we have already excluded it from our general definition of $H_\alpha(Q)$.) □

Observe that this result works by quantifying over α. This is appropriate since Rényi entropy – whether signed or unsigned – is a family of functionals, not a single functional, and so it is natural for results using Rényi entropy to refer to the full family.

By contrast, negative components in a measure do not necessarily imply negativity of signed Shannon entropy.

Example 1. *Consider the signed probability measure $Q = (2, -\frac{1}{n}, \ldots, -\frac{1}{n})$ where the term $-\frac{1}{n}$ is repeated n times. We can calculate:*

$$H_{Sh}(Q) = -2\log_2(2) - n\frac{1}{n}\log_2(\frac{1}{n}) = \log_2(n) - 2, \tag{19}$$

which is strictly positive as long as $n > 4$. To check that Rényi entropy does act as a witness in this example, let's also calculate:

$$H_{1+\epsilon}(Q) = -\frac{1}{\epsilon}\log_2[2^{1+\epsilon} + n \times (\frac{1}{n})^{1+\epsilon}] \approx -\frac{1}{\epsilon}\log_2(3), \tag{20}$$

for small positive ϵ, which is less than 0 as required by Theorem 2.

Of course, it is possible that there is another condition on our signed Shannon entropy that does witness non-classical measures. We leave this as open.

5 Doubly Stochastic Transformations

Here, we prove that if a signed measure is transformed to a second signed measure via a doubly stochastic matrix, then signed Rényi entropy is weakly increasing for all even positive integers α.

Consider signed measures P, Q viewed as column vectors in $\mathbb{R}^n$. A basic result in majorization theory (Marshall, Olkin, and Arnold, 2011, Theorem B2) says that there is a doubly stochastic matrix $\mathbf{M}$ such that $Q = \mathbf{M}P$ if and only if P majorizes Q. Recall that the second condition means:

$$\sum_{i=1}^{m} p_{(i)} \geq \sum_{i=1}^{m} q_{(i)} \text{ for all } m \leq n-1, \tag{21}$$

$$\sum_{i=1}^{n} p_{(i)} = \sum_{i=1}^{n} q_{(i)}, \tag{22}$$

where $(p_{(1)}, p_{(2)}, \ldots, p_{(n)})$ is the increasing rearrangement of P, and similarly for Q.

We will prove that signed Rényi entropy is weakly increasing under doubly stochastic transformations by showing:

Theorem 3. *If α is an even positive integer, then $H_\alpha(p_1, s-p_1, p_3, \ldots, p_n)$ is weakly decreasing in $p_1 \geq s/2$ for each fixed $s, p_3, \ldots, p_n$.*

Proof. When α is an even positive integer we can write:

$$H_\alpha(P) = -\frac{1}{\alpha - 1}\log_2\big[\frac{p_1^\alpha + (s-p_1)^\alpha + \sum_{i=3}^{n}|p_i|^\alpha}{|s + \sum_{i=3}^{n} p_i|}\big], \tag{23}$$

so that we need to show that:

$$\phi(p_1) \stackrel{\text{def}}{=} p_1^\alpha + (s-p_1)^\alpha \tag{24}$$

is weakly increasing in p_1 over the appropriate range. We have:

$$\frac{\partial \phi}{\partial p_1} = \alpha p_1^{\alpha-1} - \alpha(s-p_1)^{\alpha-1}. \tag{25}$$

If $s = 0$, then, since $\alpha - 1$ is an odd integer, it is immediate that $\partial\phi/\partial p_1 \geq 0$. Case (i): $0 < s/2 \leq p_1 < s$. Heren, since $p_1 \geq s - p_1$ and $\alpha - 1 > 0$, $\partial\phi/\partial p_1 \geq 0$. Case (ii): $0 < s \leq p_1$ or $s < 0 \leq p_1$ or $s/2 \leq p_1 < 0$. Here, since $s - p_1 \leq 0$ and $\alpha - 1$ is an odd integer, $\partial\phi/\partial p_1 \geq 0$. □

We can now conclude:

Theorem 4. *For any signed measure P and doubly stochastic matrix $\mathbf{M}$, signed Rényi entropy is weakly increasing:*

$$H_\alpha(\mathbf{M}P) \geq H_\alpha(P), \tag{26}$$

whenever α is an even positive integer.

Proof. Theorem 3 implies that $H_\alpha(P)$ is Schur-concave whenever α is an even positive integer. This follows from Remark A.2.b in Marshall, Olkin, and Arnold (2011). Inequality 26 then follows from the definition of Schur concavity and the fact that P majorizes $\mathbf{M}P$. □

This result, too, closely mirrors Koukouledikis and Jennings (2022). Their Theorem 9 states that Rényi entropy for the parametric family $\alpha = 2a/(2b-1)$, where a, b are positive integers with $a \geq b$, is Schur-concave for signed probability measures. As noted earlier, our parametric family $\alpha = 2a$, where a is a positive integer, can be obtained from a smoothness requirement on signed Rényi entropy (Brandenburger, La Mura, and Zoble, 2022).

6 Proof of Theorem 1

Lemma 6.1. *Under Axioms 0, 2′, 3, and 4, if $q \neq 0$, then $H((q)) = -\log_2(|q|)$.*

Proof. Write $h(q) = H((q))$. Axioms 0 and $2'$ imply that h is real-valued and continuous except at 0. Axiom 4 implies that $h(pq) = h(p) + h(q)$ whenever $p, q \neq 0$. This is a version of Cauchy's logarithmic functional equation, with general solution $h(q) = c \log_2 |q|$, where c is a real constant. See Theorem 3 in Aczél and Dhombres (1989, p.27). Our Axiom 3 fixes $c = -1$. □

In fact, to obtain $h(q) = -\log_2 |q|$, we can assume less than Axiom $2'$. See Corollary 2 and the development on pp.26-27 of Aczél and Dhombres (1989).

Lemma 6.2. *Under Lemma 6.1 and Axioms 4 and 5, we have $g(x) = -dx+e$ (linear) or $g(x) = d2^{(1-\alpha)x}+e$ (exponential), where $d \neq 0$, e, and $\alpha \neq 1$ are arbitrary constants.*

Proof. We adapt the argument in Daróczy (1963) to signed measures Q. From Lemma 6.1 and induction on Axiom 5, we obtain:

$$H(Q) = H((q_1) \cup \cdots \cup (q_n)) = g^{-1}\Big[\frac{\sum_j w((q_j))g(H((q_j)))}{w((q_1) \cup \cdots \cup (q_n))}\Big] = g^{-1}\Big[\frac{\sum_j |q_j| g(-\log_2 |q_j|)}{|\sum_j q_j|}\Big]. \tag{27}$$

From this and Axiom 4, we have for signed measures P and Q, provided $\sum_{i,j} p_i q_j \neq 0$:

$$g^{-1}\Big[\frac{\sum_{i,j} |p_i q_j| g(-\log_2 |p_i q_j|)}{|\sum_{i,j} p_i q_j|}\Big] = g^{-1}\Big[\frac{\sum_i |p_i| g(-\log_2 |p_i|)}{|\sum_i p_i|}\Big] + g^{-1}\Big[\frac{\sum_j |q_j| g(-\log_2 |q_j|)}{|\sum_j q_j|}\Big]. \tag{28}$$

Define $f : \mathbb{R}_{++} \to \mathbb{R}$ by $f(t) = g(-\log_2 t)$. Substituting, we get:

$$f^{-1}\Big[\frac{\sum_{i,j} |p_i q_j| f(|p_i q_j|)}{|\sum_{i,j} p_i q_j|}\Big] = f^{-1}\Big[\frac{\sum_i |p_i| f(|p_i|)}{|\sum_i p_i|}\Big] \times f^{-1}\Big[\frac{\sum_j |q_j| f(|q_j|)}{|\sum_j q_j|}\Big]. \tag{29}$$

Setting $Q = (q)$ (where $q \neq 0$), this becomes:

$$\frac{1}{|q|} f^{-1}\Big[\frac{\sum_i |p_i| f(|p_i q|)}{|\sum_i p_i|}\Big] = f^{-1}\Big[\frac{\sum_i |p_i| f(|p_i|)}{|\sum_i p_i|}\Big]. \tag{30}$$

Define $h_q : \mathbb{R}_{++} \to \mathbb{R}$ by $h_q(t) = f(|q|t)$. Then:

$$h_q^{-1}\Big[\frac{\sum_i |p_i| h_q(|p_i|)}{|\sum_i p_i|}\Big] = f^{-1}\Big[\frac{\sum_i |p_i| f(|p_i|)}{|\sum_i p_i|}\Big]. \tag{31}$$

This establishes that the maps h_q and f generate the same means when restricting the p_i's to be non-negative. By the necessity direction of a theorem on mean values (Theorem 83 in Hardy, Littlewood, and Pólya, 1952), this implies that:

$$h_q(t) = a(q)f(t) + b(q), \tag{32}$$

where $a(q)$ and $b(q)$ are independent of t, and $a(q) \neq 0$. Substituting, we get:

$$f(|q|t) = a(q)f(t) + b(q). \tag{33}$$

We can now return to follow Daróczy (1963) exactly and conclude that (restricting q to be non-negative) the solution to the functional equation 33 takes the form:

$$f(t) = d \log_2 t + e, \tag{34}$$

or

$$f(t) = dt^{\alpha-1} + e, \tag{35}$$

where $d \neq 0$, e, and $\alpha \neq 1$ are arbitrary constants. Recalling the definition of f, we then find that:

$$g(x) = -dx + e, \tag{36}$$

or

$$g(x) = d2^{(1-\alpha)x} + e, \tag{37}$$

as required. □

We can now complete the proof of Theorem 1. If g is linear as in Equation 36, then using Equation 27 we get:

$$-d \cdot H(Q) + e = d \cdot \frac{\sum_i |q_i| \log_2 |q_i|}{|\sum_i q_i|} + e \cdot \frac{\sum_i |q_i|}{|\sum_i q_i|}. \tag{38}$$

If g is exponential as in Equation 37, then again using Equation 27 we get:

$$d \cdot 2^{(1-\alpha)H(Q)} + e = d \cdot \frac{\sum_i |q_i|^\alpha}{|\sum_i q_i|} + e \cdot \frac{\sum_i |q_i|}{|\sum_i q_i|}. \tag{39}$$

Finally, if $\alpha < 0$, then $H((0,1))$ is unbounded (negative), violating real-valuedness (Axiom 0). Therefore $\alpha \geq 0$. If $\alpha = 0$, then $H(Q) = 1$ for all Q, violating non-constancy (Axiom 0). This completes the sufficiency direction of Theorem 1. The necessity of Axioms 0, $2'$, 3, 4, and 5 is a straightforward check.

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