

# A Co-Creation Value for Cooperative Games

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# The Symmetry Principle

Symmetry plays a central role in the cooperative theory of two-person bargaining games

Consider the potential gain two parties can create over their disagreement outcome

When these gains can be divided up in any fashion — the transferable-utility case — Nash (1950), Shapley (1953), and Kalai and Smorodinsky (1975) all assume it will be divided equally

Under the “divided cloth” rule of O’Neill (1982) and Aumann and Maschler (1985), each claimant first receives the value uncontested by the other, and the remainder is then divided in half

# *N*-Player Games

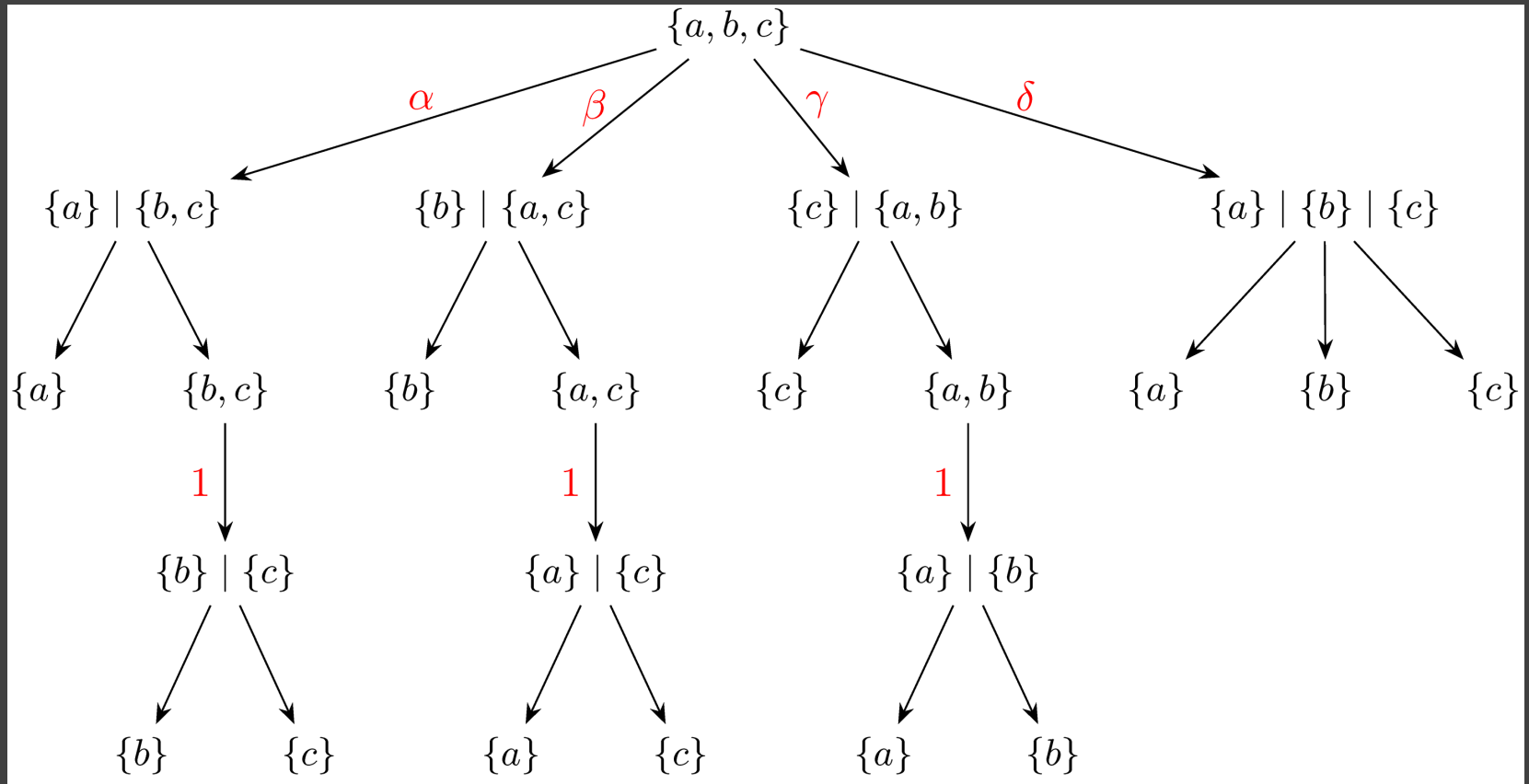
Once there are three or more players, the power of symmetry to determine the solution is greatly diminished

The basic challenge is that the disagreement point is no longer exogenous

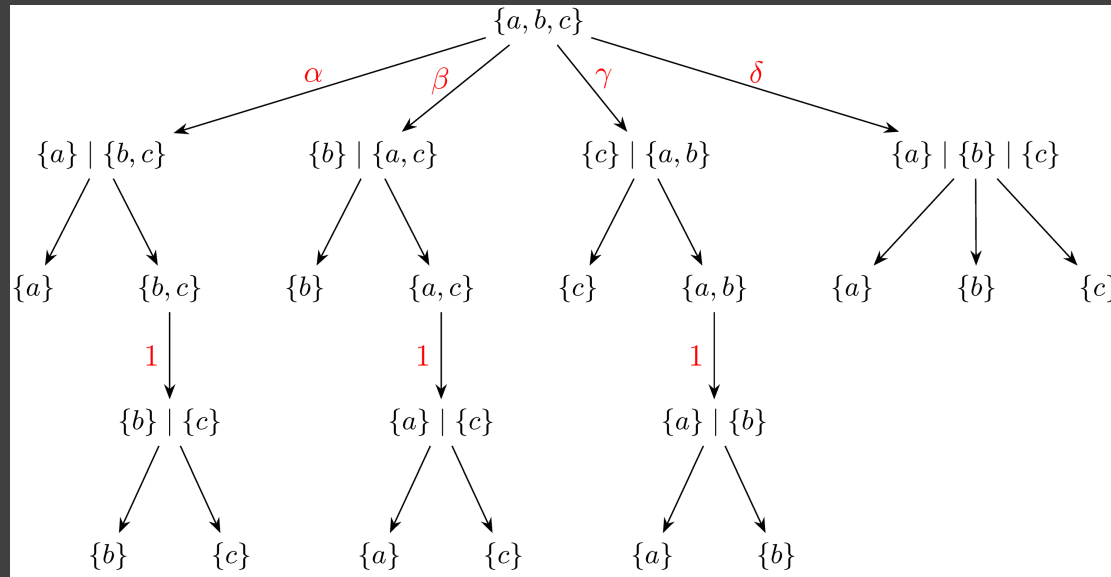
The consequence of non-agreement among three or more players depends on the way in which the multi-player coalition breaks down

There would seem to be more than one way in which this could happen

# The Subset-Refinement Game



# The CCV Solution Concept



1. The value at each subset is the expected loss on breakdown
2. All players in the subset are needed to reach agreement (symmetry returns)
3. The value at each subset is divided equally
4. Now add for each player the expected (continuation) value after breakdown
5. This defines a solution recursively

## Some Initial Properties

1. The expected payoff to player  $i$  at subset  $S$  is equal to

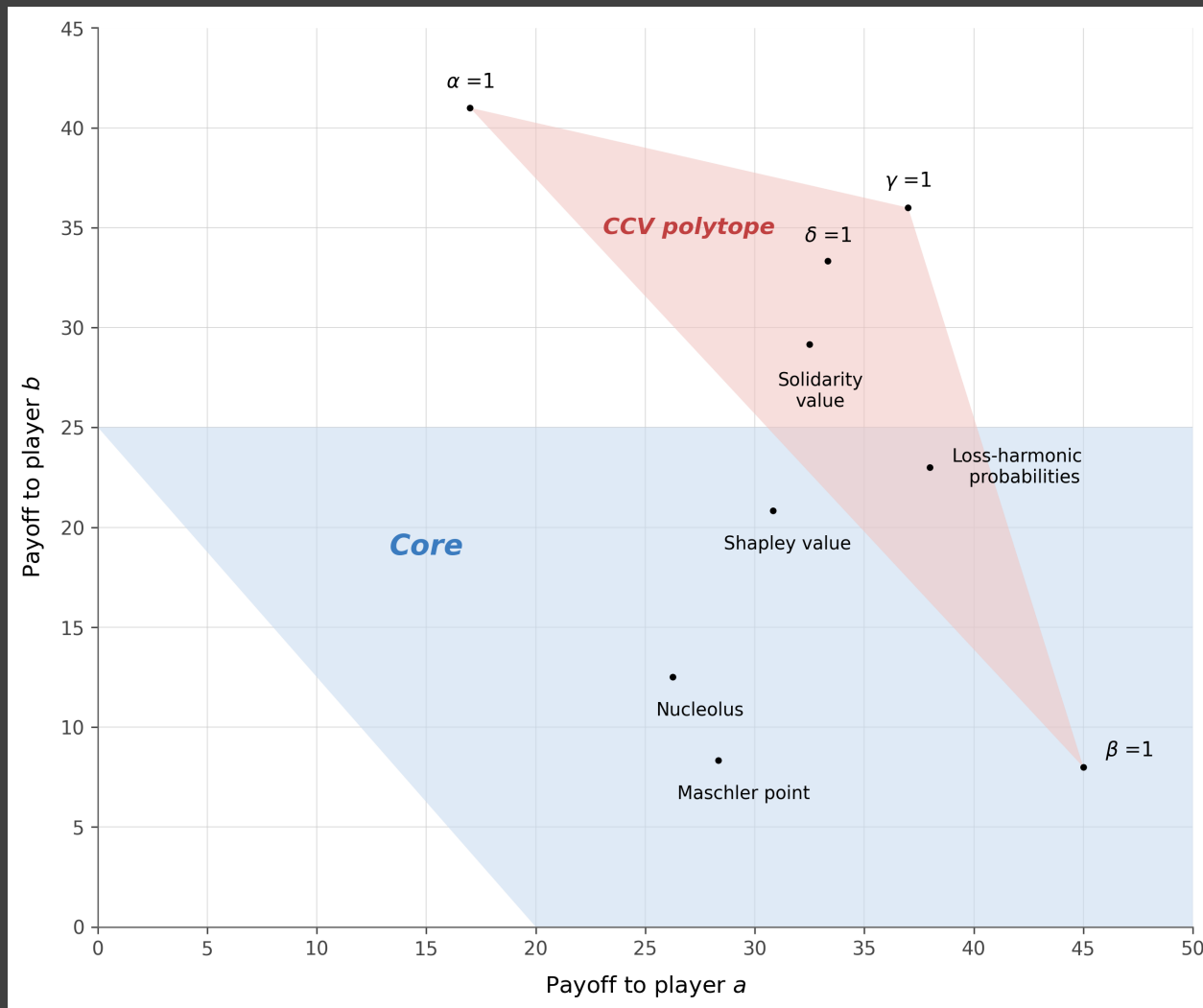
$$\phi_i(S) = \sum_{i \in T \subseteq S} \lambda_S(T) \frac{m(T)}{|T|}$$

where  $\lambda_S(T)$  is the hitting probability of  $T$ , starting from  $S$ , and  $m(T)$  is the expected loss at  $T$  on breakdown

2. The CCV is affine-invariant

3. For a fixed cooperative game  $(N, v)$ , the set of CCV expected-payoff vectors, as we vary the family of breakdown probability measures, is a polytope in  $\mathbb{R}^{|N|}$

# Relationship to Existing Solution Concepts



Nowak, A., and T. Radzik, "A Solidarity Value For n-Person Transferable-Utility Games," *International Journal of Game Theory*, 23, 1994, 43-48;  
 Sprumont, Y., "Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility," *Games and Economic Behavior*, 2, 1990,  
 378-394; Schmeidler, D., "The Nucleolus of a Characteristic Function Game," *SIAM Journal on Applied Mathematics*, 6, 1969, 1163-1170

## The Difference from the Shapley Value

Set  $\alpha = \beta = \gamma = 1/3$  and  $v(\{i\}) = 0$  for each  $i$

Write  $l_i = v(\{a, b, c\}) - v(\{j, k\})$  for each top-level local loss

The CCV is

$$\phi_i = 1/9 (l_a + l_b + l_c) + 1/6 [v(\{i, j\}) + v(\{i, k\})]$$

The Shapley value is

$$\psi_i = 1/3 l_i + 1/6 [v(\{i, j\}) + v(\{i, k\})]$$

The CCV first averages (“collectivizes”) over players the losses under breakdown and then allocates this quantity (equally)

The Shapley value first allocates (“individualizes”) each loss to the respective player and then averages this quantity over orderings (equally)

# Applications

Three assumptions on breakdown probabilities:

- (i) only partitions where one player is left out receive positive weight
- (ii) if the loss when player  $j$  is left out is at least as large as when player  $i$  is left out, the probability of the first partition is at most equal to the probability of the second partition
- (iii) if the loss is 0 when some player  $i$  is left out, then only 0-loss partitions receive positive weight and the weights on these partitions are equal

Two examples:

The Glove Game – There are  $N$  players with  $B = \alpha N$  buyers and  $S = (1 - \alpha)N$  sellers where  $1/2 < \alpha < 1$  and any pair made up of one buyer and one seller creates value of 1

The Overcapacity Game – There are two sellers, each with two units to offer at cost 0, and three buyers, each interested in one unit which they value at 1

## The Glove Game

The core gives 0 to each buyer and 1 to each seller

For any  $\alpha = 1/2 + \epsilon$ , the Shapley value gives close to \$1 to each buyer and close to \$0 to each seller as  $N \rightarrow \infty$

The CCV gives  $(1 - \alpha)/2\alpha$  to each buyer and  $1/2$  to each seller (for any  $N$ )

Evidently, the CCV yields a very different solution in this game from either the Shapley value or the core

The Shapley value smooths out the core solution but, unlike the CCV, still exhibits “extreme” behavior in the large- $N$  case

## The Overcapacity Game

The core gives 0 to each seller and 1 to each buyer

The Shapley value gives  $17/20 = 0.850$  to each seller and  $13/30 = 0.433$  to each buyer

The CCV gives  $[0.589, 0.617]$  to each seller and  $[0.589, 0.607]$  to each buyer

The core appears to underestimate the position of a seller in the game, since, without a seller, the overall value shrinks from 3 to 2

The Shapley value avoids this extreme, but goes very far in the opposite direction in giving sellers nearly double what it gives buyers

The CCV gives a strictly intermediate range of answers and its solution set is closely clustered around an equal five-way division giving  $3/5$  to each of the five players

## Additional Properties

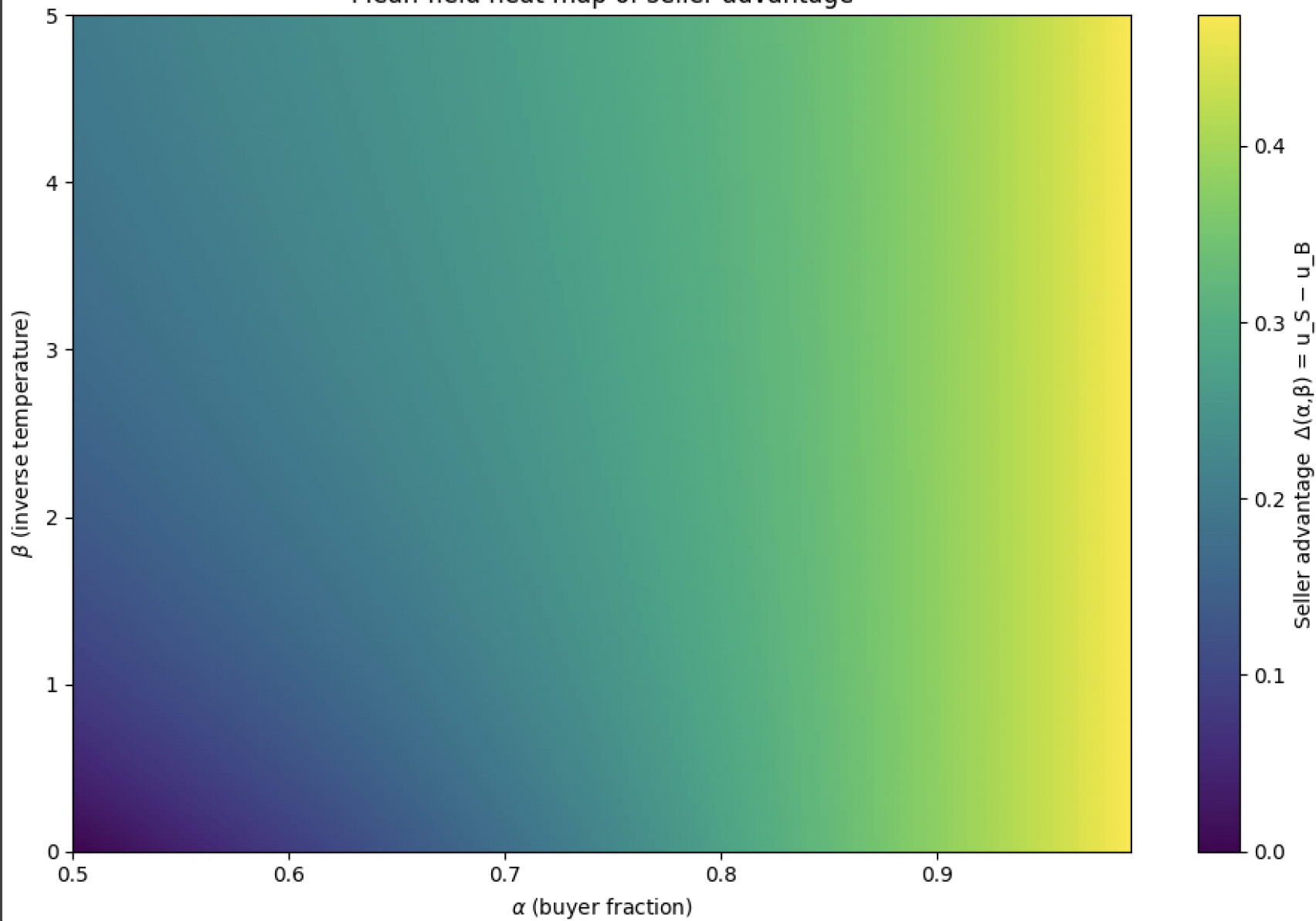
1. The CCV can be axiomatized via subset-refinement game analogs to Shapley Efficiency, Symmetry, Dummy, and Additivity
2. The Shapley value can be obtained as the CCV of a game with signed breakdown probabilities

$$p_T(\pi) = (-1)^{x_\pi} (x_\pi - 1)!$$

where  $x_\pi$  is the number of cells in the partition  $\pi$  — and in this case, all hitting probabilities are 1

3. The CCV can be extended to allow context dependency (externalities) by redefining the recursion to operate on pairs  $(T, \sigma)$  where  $T \subseteq N$  and  $\sigma$  is a partition-node that produces  $T$  in the subset-refinement graph

Mean-field heat map of seller advantage



Thank you