# Reevaluating the Shapley Value: Uniqueness of the $\alpha$-Procedure

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## Proof of Theorem 2

Our general weighting system specifies weights $w_j(S,i;k) \geq 0$ that determine each player $j$'s share of the $k$th interval of the total value $(1-\alpha)m_i$, when $i$ is the joining player.

Write $w_j^*(S,i;k)$ for the weights of the $\alpha$-procedure:

$$w_j^*(S,i;k) = \begin{cases} \dfrac{1}{|S|-k} & \text{if } j \neq i, k \leq \min\{i,j\}; \\ 1 & \text{if } i = j = k = |S|; \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

**Theorem 2.** *Under the Priority and Monotonicity Axioms, the weights $w_j^*(S,i;k)$ of the $\alpha$-procedure uniquely yield the Shapley value.*

To begin the proof, recall from Theorem 1 and Corollary 1.1 that the $\alpha$-procedure leads to the Shapley value. Also, these weights satisfy the Priority and Monotonicity Axioms. Therefore, we need to prove that no other weighting system leads to the Shapley value. For any fixed $\alpha$, the expected payoffs to the players are an $\alpha : 1-\alpha$ weighted average of the Shapley value and the $\alpha = 0$ solution. Thus, we only need to establish necessity of the weights in Equation A.1 for the case $\alpha = 0$.

We first suppose that $S = N$. The proof proceeds in two steps. In Step 1, we find conditions on the weights so that the expected payoffs are the Shapley values. In Step 2, we show that imposing the Priority and Monotonicity Axioms reduces these weights to those given by Equation A.1.

*Step 1:* Consider the Unanimity Game, i.e., the game where $v(N) > 0$ and $v(S) = 0$ for $S \subsetneq N$. Note that the characteristic function is 0 everywhere in any subgame. The Shapley value is $v(N)/|N|$ for all players. In this game, there is only one non-zero interval of value, namely, $[0, m_1(N)] = [0, v(N)]$. The expected payoff to player $j$ is therefore given by (recalling $\alpha = 0$):

$$\pi^j(N;0) = \frac{1}{|N|}\sum_{i\in N}\pi_j(N,i;0) = \frac{1}{|N|}\sum_{i\in N}\sum_{k=1}^{1} w_j(N,i;k)(m_k - m_{k-1}) = \frac{m_1}{|N|}\sum_{i\in N} w_j(N,i;1). \tag{A.2}$$

The payoffs in all subgames are 0. Therefore, if player $j$ is to receive its Shapley value, it is necessary that:

$$\sum_{i\in N} w_j(N,i;1) = 1 \text{ for all } j \in N. \tag{A.3}$$

Also, since weights sum to 1, we must also gave:

$$\sum_{j\in N} w_j(N,i;1) = 1 \text{ for all } i \in N. \tag{A.4}$$

We refer to Equation A.3 as the **Shapley condition** and to Equation A.4 as the **Sum condition**.

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*Step 2:* We now make use of the Priority and Monotonicity Axioms. The case $|N| = 1$ is trivial. Both the Shapley and Sum conditions imply $w_1(N, 1; 1) = 1$, which is exactly Equation A.1 when $|N| = 1$. So, we assume $|N| \geq 2$. Priority implies $w_j(N, j; 1) = 0$, so that the Shapley condition becomes:

$$\sum_{i \in N \setminus j} w_j(N, i; 1) = 1 \text{ for all } j \in N, \tag{A.5}$$

and the Sum condition becomes:

$$\sum_{j \in N \setminus i} w_j(N, i; 1) = 1 \text{ for all } j \in N. \tag{A.6}$$

We now alternate between applying the Sum condition and the Shapley condition. Monotonicity and the Sum condition imply:

$$(|N| - 1)w_1(N, i; 1) \leq \sum_{j \in N \setminus i} w_j(N, i; 1) = 1 \text{ for } i \neq 1, \tag{A.7}$$

from which:

$$w_1(N, i; 1) \leq \frac{1}{|N| - 1} \text{ for } i \neq 1. \tag{A.8}$$

If $w_1(N, i; 1) < 1/(|N| - 1)$ for any $i \neq 1$, then:

$$\sum_{i \in N \setminus 1} w_1(N, i; 1) < 1, \tag{A.9}$$

which violates the Shapley condition A.5. Therefore, we must have:

$$w_1(N, i; 1) = \frac{1}{|N| - 1} \text{ for } i \neq 1. \tag{A.10}$$

By Monotonicity and Equation A.10:

$$w_j(N, i; 1) \geq w_1(N, i; 1) = 1/(|N| - 1) \text{ for } i \notin \{1, j\}. \tag{A.11}$$

Using the Sum condition A.6, we conclude that:

$$w_j(N, i; 1) = \frac{1}{|N| - 1} \text{ for all } j \in N \text{ and } i \notin \{1, j\}. \tag{A.12}$$

The preceding argument covers all weights in the first interval of value, excepting the weights $w_j(N, 1; 1)$ for $j \geq 2$. Returning to the Shapley condition A.5, we can write:

$$\sum_{i \in N \setminus j} w_j(N, i; 1) = w_j(N, 1; 1) + \frac{|N| - 2}{|N| - 1} = 1 \text{ for } j \geq 2, \tag{A.13}$$

which implies:

$$w_j(N, 1; 1) = \frac{1}{|N| - 1} \text{ for } j \geq 2. \tag{A.14}$$

We have now established that, under Priority and Monotonicity, weights that lead to the Shapley value must satisfy:

$$w_j(N, i; k) = \begin{cases} \dfrac{1}{|N| - k} & \text{if } j \neq i, k = 1; \\ 1 & \text{if } i = j = k = |N|, k = 1. \end{cases} \tag{A.15}$$

Equation A.15 coincides with Equation A.1 when $k = 1$. It remains to determine necessary conditions on weights for $k \geq 2$. To do so, consider a game in which player 1's only (strictly) positive marginal contribution is $m_1(N)$; its marginal contribution to all other coalitions is 0. Thus, player 1's Shapley value for this game is $m_1/|N|$. Suppose that the marginal contributions $m_j(N)$ for all other players $j \neq 1$ are all larger than $m_1(N)$ and are all distinct. Assume player 1's payoff is the Shapley value when $\alpha = 0$.

We established above that player 1's payoff from the first interval of value is $m_1/|N|$. It follows that the weights for player 1 from all higher intervals must be 0, i.e., $w_1(N, i; k) = 0$ for $k \geq 2$. This completes the argument that the weights $w_1(N, i; k)$, for all $i$ and $k$, are equal to those in Equation A.1. This is for player $j = 1$.

We now repeat this exercise by considering games in which player 1 is a dummy player. Analogously to above, we consider the Unanimity Game (strictly, subgame) among players $2, \ldots, N$. If $|N| = 2$, Priority implies $w_2(N, 2, 2) = 1$. For $N \geq 3$, we repeat our earlier argument to obtain the analog to Equation A.14:

$$w_j(N, i, 2) = \frac{1}{|N| - 2} \text{ for } j \neq i, j \geq 2. \tag{A.16}$$

As before, we next consider a game in which player 1 is a dummy player, player 2's only positive marginal contribution is $m_2/|N|$, and the marginal contributions for all other players are all larger than $m_2(N)$ and are all distinct. Player 2's Shapley value for this game is $m_2/|N|$. Assume player 2's payoff is also the Shapley value when $\alpha = 0$. Then, Equation A.16 implies that player 2's payoff from the second interval is $m_2/|N|$. It follows that the weights for player 2 from all higher intervals must be 0, i.e., $w_2(N, i; k) = 0$ for $k \geq 3$. This completes the argument that the weights $w_2(N, i; k)$, for all $i$ and $k$, are equal to those in Equation A.1. This is for player $j = 2$.

We next move to player 3, by making both players 1 and 2 dummy players. In this manner, we establish that the weights $w_j(N, i; k)$, for all $i$, $k$, and $j$, are equal to those in Equation A.1

*Continuation:* We can re-run the preceding argument for any $S \subsetneq N$, thereby completing the proof of Theorem 3.