Reevaluating the Shapley Value:
The NTU Case*

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We show how our α-procedure introduced in Brandenburger and Nalebuff (2024) can be extended to analyze NTU games. For $\alpha = 1$, this problem has been addressed in Hart and Mas-Colell (1996), who provide a procedure that yields the consistent Shapley value Maschler and Owen (1989, 1992). When $0 \leq \alpha < 1$, a new procedure is required, just as in the TU case, since we need a rule for allocating the $1 - \alpha$ fraction of value. We achieve this by defining the NTU marginal contribution of a player to a coalition S, which leads to our generalized procedure.

Given an NTU game (N, V), we assume that the feasible sets $V(S)$ satisfy the standard conditions on the characteristic function; see, in particular, conditions (A.1)-(A.3) in Hart and Mas-Colell (1996). Let $\partial V(S)$ denote the boundary of the feasible set for S. For convenience, we perform two normalizations. We set $\partial V(\{i\}) = 0$ for all i. We also scale the utilities for all players so that the maximum feasible utility level of each player i in $V(N)$ is 1.

We begin with the case where $\partial V(N)$ is a hyperplane (therefore, the unit simplex under our scaling) and then show how to extend our analysis to the general convex case as in Maschler and Owen (1989, 1992). Let Ψ denote the tuple of payoffs from the procedure. Our normalization implies $\Psi(i) = 0$ for one-player games.

Assume inductively that we have a solution for coalitions of size up to $|N| - 1$ (for any characteristic function). Fix a game (N, V). We derive the solution for the set N. The marginal contributions associated with N are defined by:

$$d^i(N) = \max\{c^i : (c^i, \Psi(N \backslash i)) \in V(N)\}, \tag{B.1}$$

This quantity is the maximum possible payoff to player i given that the other players obtain their payoffs in the game without i. Our definition of marginal contribution differs from that in Hart and Mas-Colell (1996) in two ways. First, the marginal contributions as defined in Equation B.1 are independent of the order in which players arrive. In Hart and Mas-Colell (1996), the marginal contributions are defined inductively based on a specific ordering of player arrivals. Second, outside of a hyperplane game—that, is, a game where $V(S)$, for each $S \subseteq N$, is a half space—the solution to the subgame $\Psi(N \backslash i)$ need not be the average marginal contribution of each player in that game.

With our set of marginal contributions, and following our earlier convention, we index the players in order of increasing marginal contribution. The inductive step in our NTU procedure is obtained by adapting our earlier TU game.

From the set N we randomly select a player to be at risk. Given player i is at risk, we assign the probability τ_{ji} that player j is the proposer as in Equation (5) in Brandenburger and Nalebuff (2024), substituting the marginal contribution $d^i(S)$ for each $m_i(S)$, and then dividing by $d^i(S)$:[1]

$$\tau_{ji}(S) = \begin{cases} \frac{(1-\alpha)}{d^i} \displaystyle\sum_{k=1}^{\min\{i,j\}} \frac{d^k - d^{k-1}}{|S| - k} & \text{if } j \neq i; \\ \alpha & \text{if } j = i < |S|; \\ \alpha + \frac{(1-\alpha)}{d^{|S|}}(d^{|S|} - d^{|S|-1}) & \text{if } j = i = |S|. \end{cases} \tag{B.2}$$

In this way, the bargaining parameter α enters our NTU-procedure.

The procedure, contingent on the random selection of i and j, assigns to everyone their value in $\Psi(N \backslash i)$, where the proposer i receives an additional amount $d^i(N)$. (As before, we extend $\Psi(N \backslash i)$ so that player

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[1] If some of the $d^i(S)$ are equal, we may be adding some extra intervals of length 0.

i, who is not a member of $N\backslash i$, receives 0 under $\Psi(N\backslash i)$.) Because $\partial V(N)$ is a hyperplane (normalized to the unit simplex), it is always efficient and feasible to assign $d^i(N)$ to the player making the proposal. The payoffs $\Psi(N)$ are the expected values, where each player has an equal chance of being at risk:

$$\Psi(N) = \frac{1}{|N|} \sum_{i=1}^{N} \left[\Psi(N\backslash i) + \sum_{j=1}^{N} \tau_{ji} d^i(N\backslash i) e^j\right], \tag{B.3}$$

where e^j is the jth unit vector. Again, because $\partial V(N)$ is a hyperplane, this expected value is both efficient and feasible. Moreover, the same argument as in the proof of Theorem 1 of Brandenburger and Nalebuff (2024) establishes:

$$\frac{1}{|N|} \sum_{i=1}^{N} \tau_{ji} d^i(N\backslash i) = d^j(N\backslash i), \tag{B.4}$$

from which:

$$\Psi(N) = d(N) + \frac{1}{|N|} \sum_{i=1}^{N} \Psi(N\backslash i), \tag{B.5}$$

where $d(N)$ is the tuple made up of the $d^j(N)$'s. We can see that this procedural formula is the NTU analog to the Shapley recursion formula.

This is the solution for the case where $\partial V(N)$ is a hyperplane. To find the solution for general $V(N)$, we look for a fixed point as in Maschler and Owen (1992). Start with a point p in the unit simplex. Consider the ray from the origin through p. This ray will intersect $\partial V(N)$ at some point q. Let hyperplane $H(q)$ be tangent to $\partial V(N)$ at q. Normalize $H(q)$ so that it is the unit simplex, and apply the same scaling to $V(N)$.

Now consider the game where the scaled $V(N)$ is extended to $H(q)$. Here, the boundary is a hyperplane, so we can apply the solution for $\Psi(N)$ from Equation B.5. This defines a continuous mapping from the unit simplex to itself, namely, from p to q to $\Psi(N)$. This mapping therefore has a fixed point. Moreover, the fixed point is a tangency point and thus lies on the boundary of (the scaled) $V(N)$. At the fixed point, the solution for $\Psi(N)$ is then defined as the consistent solution to the feasible set $V(N)$.

The intuition for selecting the fixed point is similar to that for the Independence of Irrelevant Alternatives Axiom of decision theory: $\Psi(N)$ is a solution for a larger set that includes $V(N)$ and it remains feasible in the smaller set $V(N)$, so we require it to be the solution for the smaller set.

Observe that the inductive step has two parts. We start with $|N|$ players and randomly break the set into $|N|-1$ inside players and one at-risk player. We apply the procedure to a game with $|N|-1$ players, and we divide up the at-risk player's marginal contribution to obtain the solution to a game with $|N|$ players. This first step is carried out when the boundary for $V(N)$ is a hyperplane. We then use the solution to all such games to find a fixed point for general $V(N)$. This approach is similar to the way the Nash bargaining solution (Nash, 1950) is constructed.

We offer some remarks on our NTU procedure. First, if the game is TU, the procedure leads to the same outcome as under our α-procedure in Brandenburger and Nalebuff (2024). Next, for two-person games, our NTU procedure leads to the Nash (1950) bargaining solution for all values of α. Indeed, when the boundary of the bargaining set is a line, the NTU procedure selects the midpoint $\Psi = 1/2[(\alpha, 1-\alpha) + (1-\alpha, \alpha)] = (1/2, 1/2)$. As in the Nash bargaining solution, the NTU procedure for convex sets selects the boundary point that is the midpoint of the tangent line at that boundary point. For $\alpha = 1$, our procedure leads to the same consistent solution(s) as in Hart and Mas-Colell (1996).[2] Any consistent solution is based on the solution to a hyperplane game, and our procedures coincide in hyperplane games when $\alpha = 1$.

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[2] Hart and Mas-Colell allow for a penalty in the case of disagreement. Our solutions coincide when the penalty in Hart and Mas-Colell is 0.