Introduction to Game Theory:

Cooperative Game Theory

NYU Shanghai

Spring 2017

The Two Branches of Game Theory

In **non-cooperative game theory**, a game model is a detailed description of all the moves available to the players (the matrix or the tree)

In **cooperative game theory**, a game model abstracts away from this level of detail and describes only the outcomes that result when players come together in different combinations

The terms are misleading!

Non-cooperative theory can study cooperation --- e.g., in the theory of repeated games

Cooperative theory can study competition --- e.g., in the theory of the core

Better (but non-standard) terms would be **procedural game theory** and **combinatorial game theory**

Another Way to Say It …

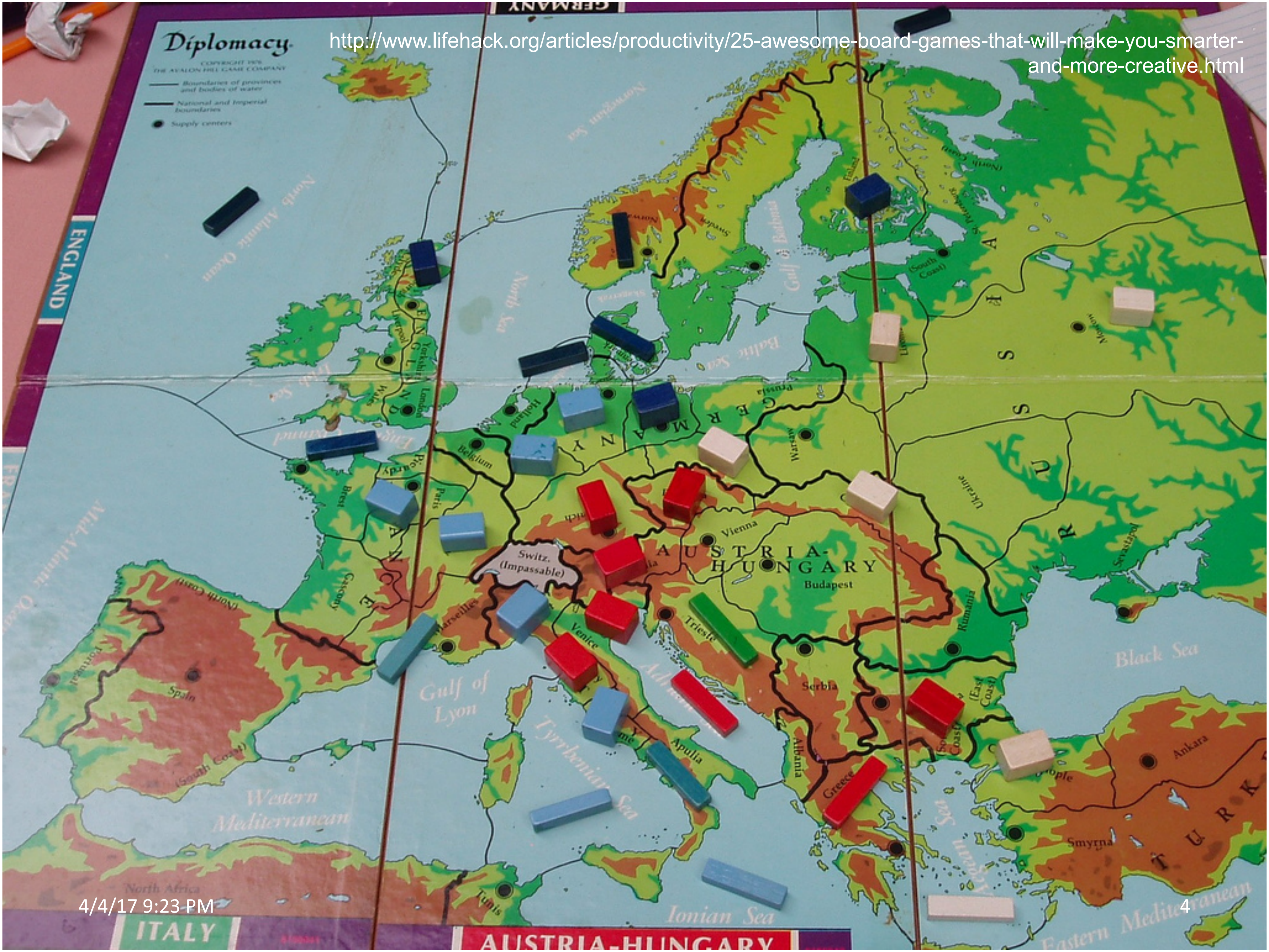
Non-cooperative theory studies **individual action** focused on **individual interests**

Cooperative theory studies **joint action** focused on **joint interests**

But it is not useful to spend too long on interpretation at this stage

Let's see some cooperative theory in action …

http://www.lifehack.org/articles/productivity/25-awesome-board-games-that-will-make-you-smarter-and-more-creative.html



Definition of a Cooperative Game

A **cooperative game** consists of

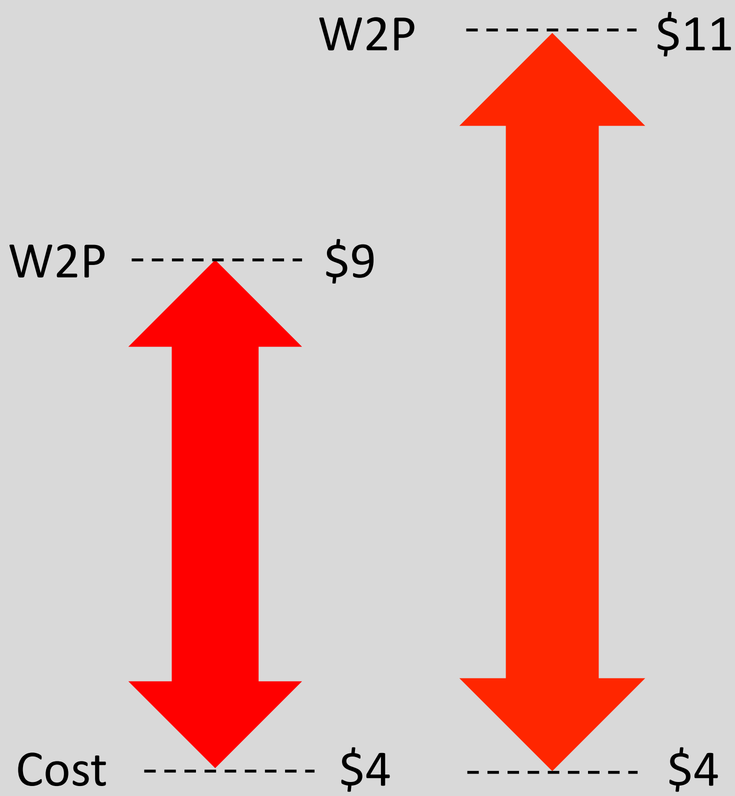
a **set of players** $N = \{1, 2, \ldots, n\}$

a **characteristic function** $v: 2^N \to \mathbb{R}$

where 2^N denotes the set of all subsets of N and $\mathbb{R}$ denotes the real numbers

For each subset S of N the number $v(S)$ is interpreted as the value created when the members of S come together and interact

Cooperative Games: Example #1



Player 1 is a seller with one unit to sell (cost $4)

Player 2 is a buyer interested in one unit (willingness-to-pay $9)

Player 3 is a buyer interested in one unit (willingness-to-pay $11)

$N = \{1,2,3\}$
$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$
$v(\{1,2\}) = 5, v(\{1,3\}) = 7, v(\{2,3\}) = 0$
$v(\{1,2,3\}) = 7$

Division of Value

Given a cooperative game (N, v), the quantity $v(N)$ specifies the overall amount of value created

We can then ask how this overall value is divided up among the various players

Intuition says that bargaining among the players in the game determines the division of overall value

Intuition also says that a player's 'power' in this bargaining depends on the extent to which the player needs other players to create value, as compared with the extent to which other players need this player

Marginal Contribution

Given the set of players N and a particular player i, let $N \backslash \{i\}$ denote the subset of N consisting of all the players except player i

The **marginal contribution** of player i is $v(N) - v(N \backslash \{i\})$, to be denoted by MC_i

In words, the marginal contribution of a particular player is the amount by which the overall value created would change if the player in question were to leave the game

Example #1 cont'd: $\mathrm{MC}_1 = ?$, $\mathrm{MC}_2 = ?$, $\mathrm{MC}_3 = ?$

A Marginal Contribution Principle

An **allocation** is a collection $(x_1, x_2, \dots, x_n)$ of numbers

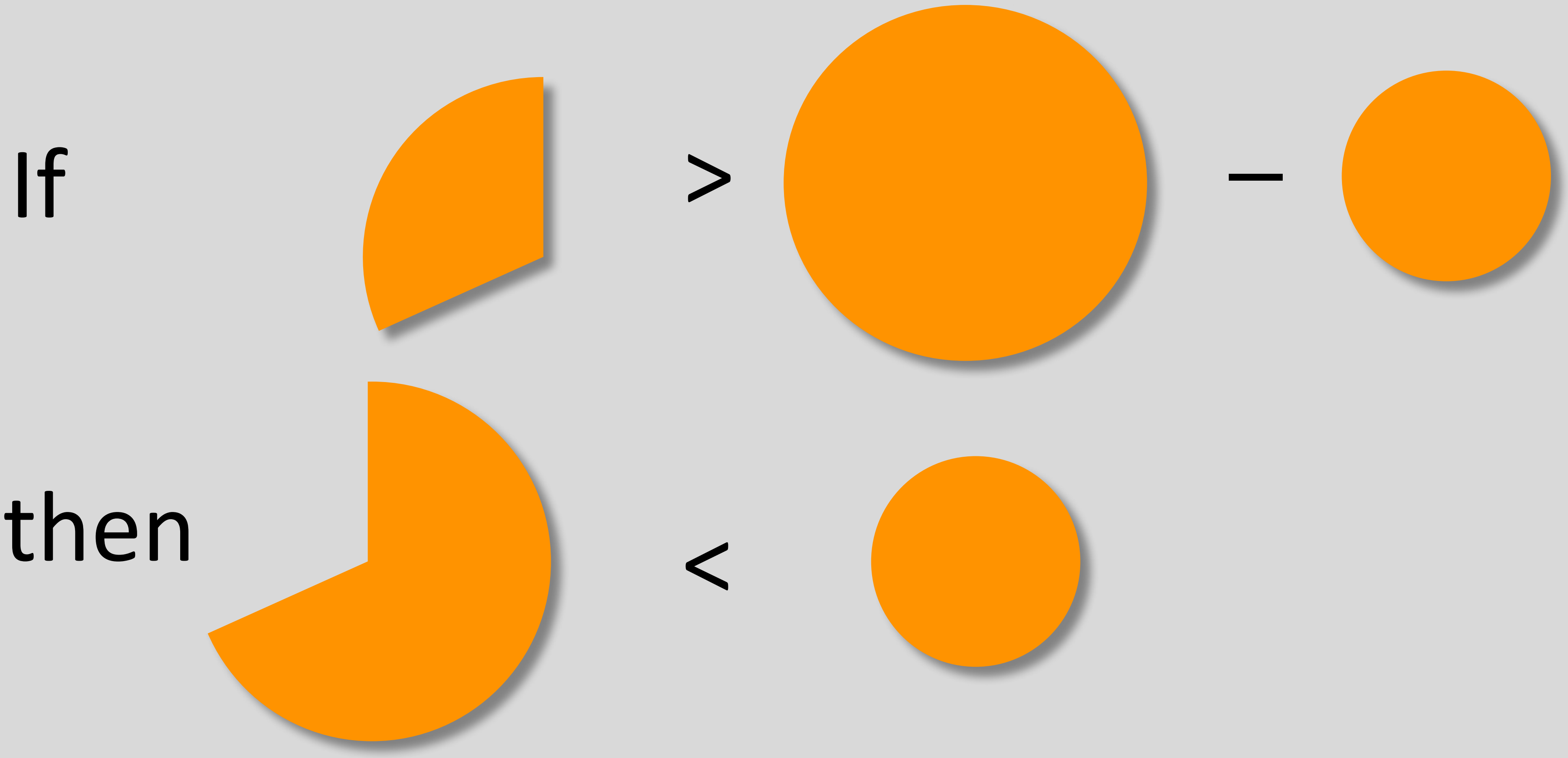
Here, the quantity x_i denotes the value received by player i

An allocation $(x_1, x_2, \dots, x_n)$ is **individually rational** if $x_i \geq v(\{i\})$ for all i

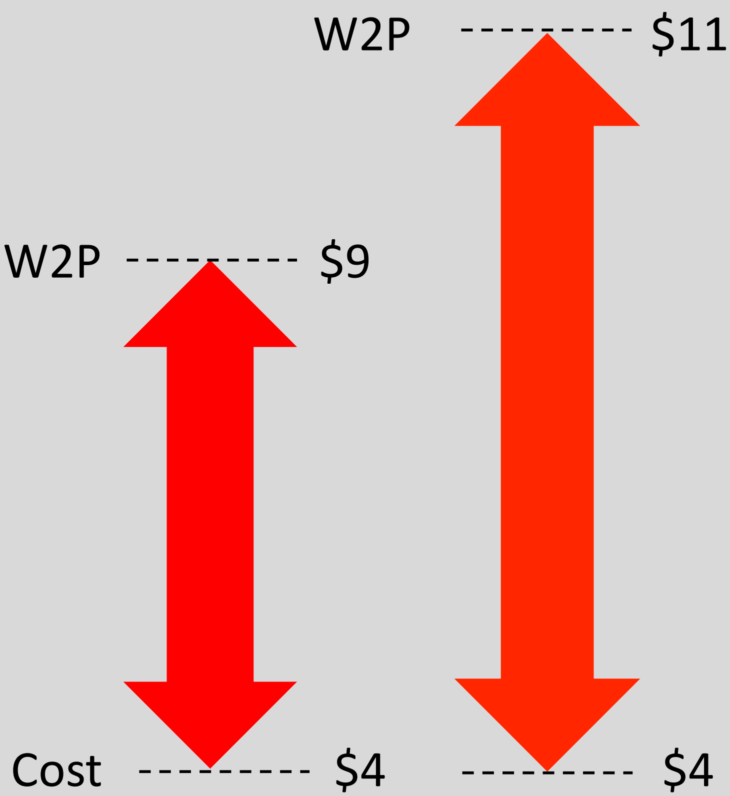
An allocation $(x_1, x_2, \dots, x_n)$ is **efficient** if $\sum_{i=1}^{n} x_i = v(N)$

An (individually rational and efficient) allocation $(x_1, x_2, \dots, x_n)$ satisfies the **Marginal Contribution Principle** if $x_i \leq \mathrm{MC}_i$ for all i

Argument for this Marginal Contribution Principle



Example #1 cont'd



Player 1 is a seller with one unit to sell (cost $4)

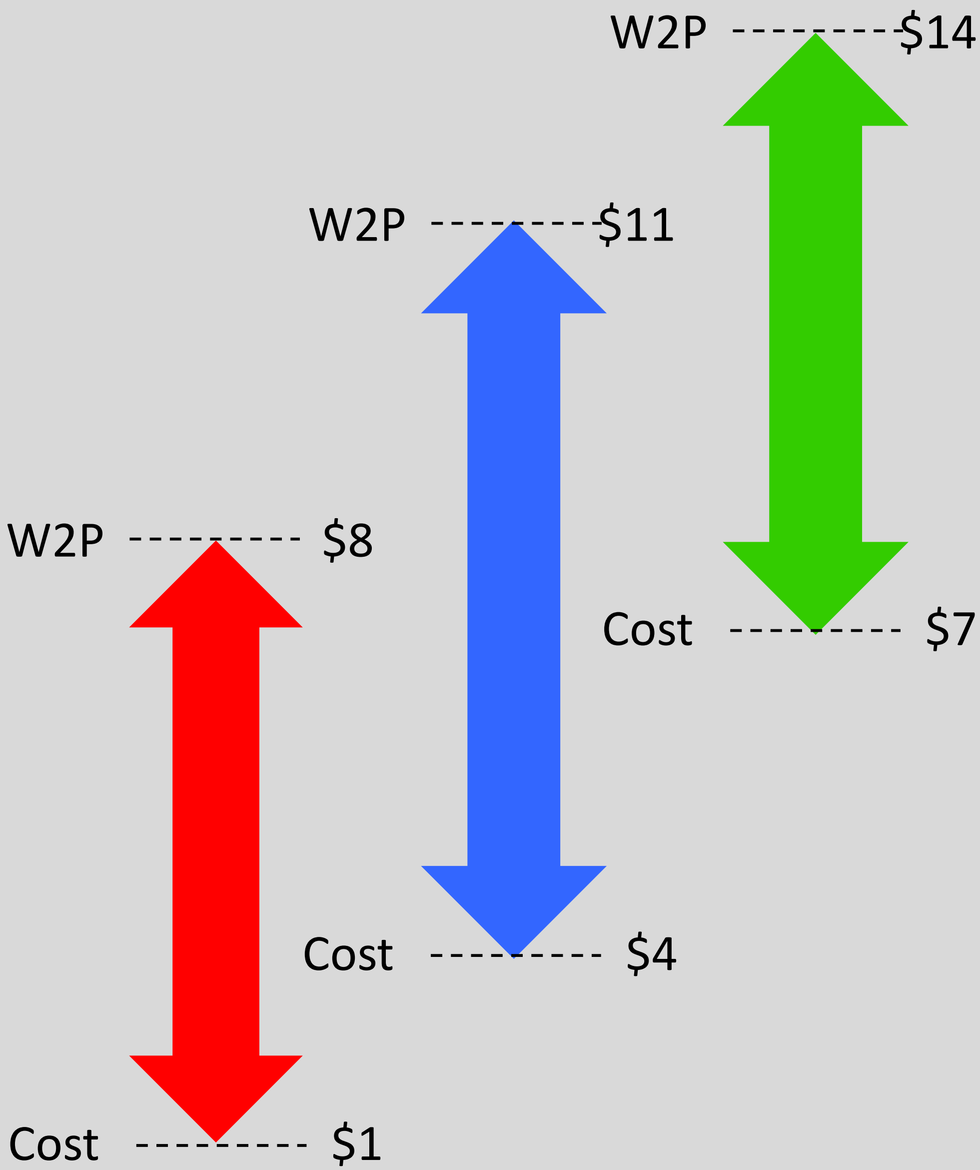
Player 2 is a buyer interested in one unit (willingness-to-pay $9)

Player 3 is a buyer interested in one unit (willingness-to-pay $11)

$N = \{1,2,3\}$
$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$
$v(\{1,2\}) = 5, v(\{1,3\}) = 7, v(\{2,3\}) = 0$
$v(\{1,2,3\}) = 7$

What does the Marginal Contribution Principle say about how the overall value of $7 gets divided among the players?

Cooperative Games: Example #2



There are three firms, each with one unit to sell

There are two identical buyers, each interested in one unit of product from some firm

The blue firm can spend $1 to raise W2P to $12 and lower Cost to $3

The Core

An allocation $(x_1, x_2, \ldots, x_n)$ is in the **core** of the game if it is efficient and is such that for every subset S of N we have

$$\sum_{i \in S} x_i \geq v(\{S\})$$

The **marginal contribution** of subset S of N is $v(N) - v(N \backslash S)$, to be denoted by MC_S

Theorem: An efficient allocation $(x_1, x_2, \ldots, x_n)$ lies in the core if and only if for every subset S of N we have

$$\sum_{i \in S} x_i \leq \mathrm{MC}_S$$

This shows that the core is a strengthening of the Marginal Contribution Principle

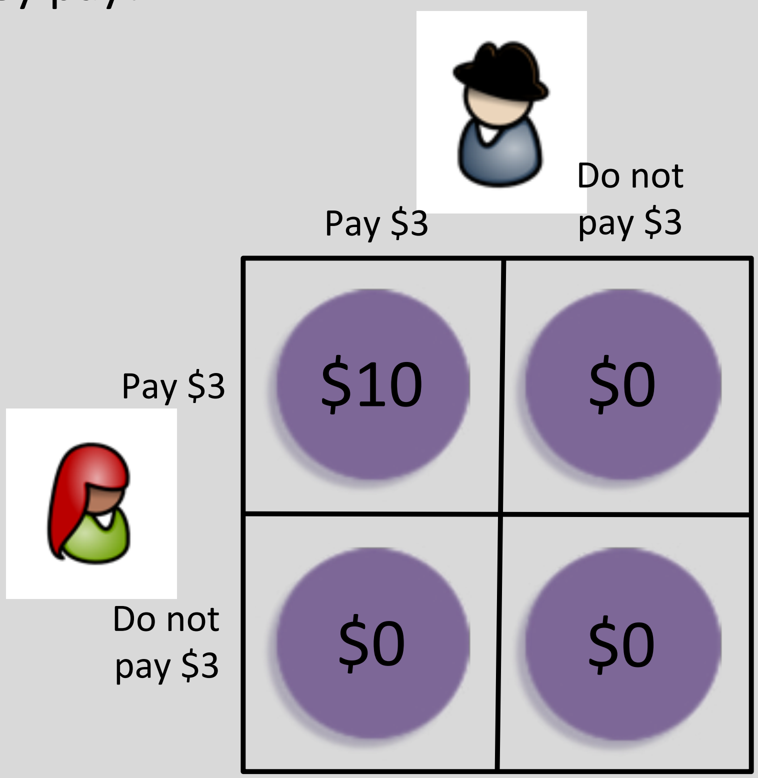
Example #3: There are two sellers, each with two units to sell where Cost = $0. There are three buyers, each interested in buying one unit where W2P = $1 for either seller's product.

Example adapted from Postlewaite, A., and R. Rosenthal, "Disadvantageous Syndicates," *Journal of Economic Theory*, 9, 1974, 324-326

Economic Efficiency from the Perspective of Game Theory

Efficiency (resp. inefficiency) arises when the pie that results from the strategic moves that the players choose is (resp. is not) the largest possible

Example: Alice and Bob each has to decide whether or not to pay $3 to a third party. If they both pay $3, they are in a position to transact with each other and together create $10 of value. Will they pay?



Brandenburger, A., and H. Stuart, “Biform Games,” *Management Science*, 53, 2007, 537-549

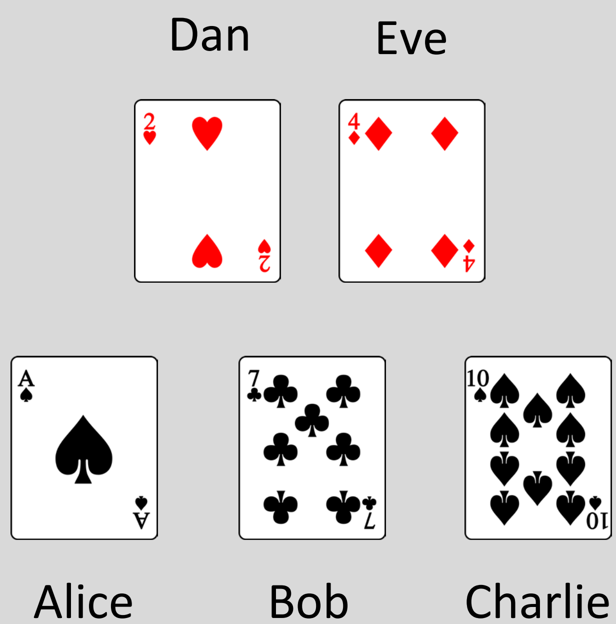
No Bargaining Problems

A game exhibits **No Bargaining Problems** if the sum, over all the players, of each player's marginal contribution is equal to the total value of the game:

$$\sum_{i=1}^{n} \mathrm{MC}_i = v(N)$$

Economic Efficiency Cont'd

Example: Alice, Bob, and Charlie each hold a black playing card. Dan and Eve each hold a red playing card. Any black card and red card together are worth $100. Before negotiations among the players begin, Alice can dog-ear the black cards that Bob and Charlie get. A red card and a dog-eared black card are together worth only $50. Will she do so?

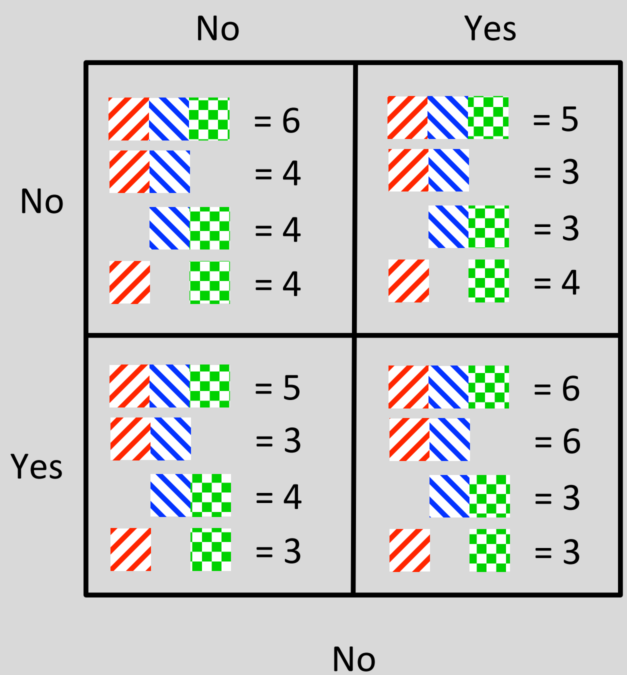


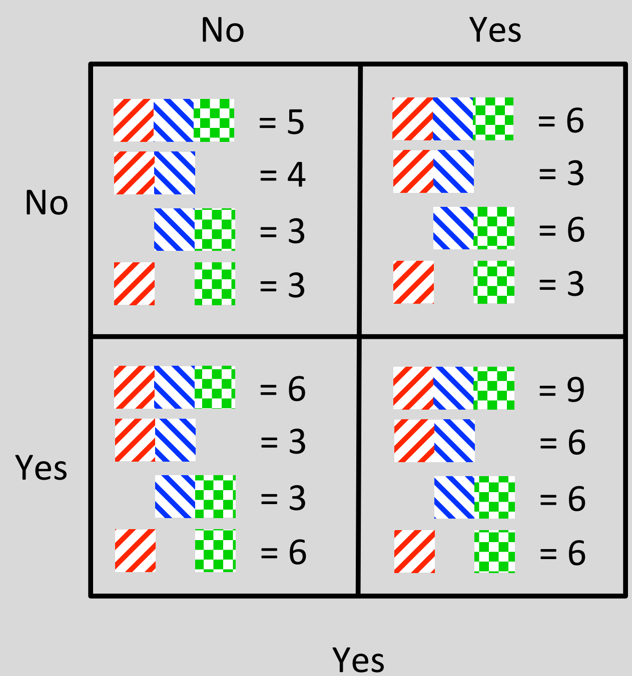
No Externality Problems

A game exhibits **No Externality Problems** if, taking each player i at a time (and holding constant the pre-bargaining choices made by all the players other than i), the choices i makes do not affect the pie created by all the players other than i

Economic Efficiency Cont’d

Example: Alice chooses the row, Bob chooses the column, and Charlie chooses the matrix. Will Ann, Bob, Charlie choose Yes-Yes-Yes?





	No	Yes
No	2, 2, 2	2, 1, 2
Yes	1, 2, 2	3, 3, 0

No

	No	Yes
No	2, 2, 1	0, 3, 3
Yes	3, 0, 3	3, 3, 3

Yes

No Coordination Problems

A game exhibits **No Coordination Problems** if the maximum of the overall pie can be found by maximizing the overall pie player-by-player

A Theorem on Efficiency

If a game exhibits No Bargaining Problems, No Externality Problems, and No Coordination Problems, then each player has a dominant strategy and, when these strategies are played, the largest overall pie is created.

We see that for efficiency to be assured, various kinds of interdependencies among players have to be ruled out.