Introduction to Game Theory:

Two-Player Zero-Sum Games

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Two-Player Zero-Sum Games

Bob's strategies

Ann's strategies

	b_1	•••	b_j	•••
a_1				
⋮				
a_i			$\pi^B(b_j, a_i)$ $\pi^A(a_i, b_j)$	
⋮				

Two-Player Zero-Sum Games

A two-player matrix game is called **zero-sum** if for every cell in the matrix

$$\pi^A(a_i, b_j) + \pi^B(b_j, a_i) = 0$$

Or, the payoffs could sum to some other constant number across cells (sometimes, these games are called **constant-sum**)

It is a convenient normalization to use 0

Zero-sum games can be seen as ‘purely competitive’ in that if one cell is better than another cell for one player, it must be worse for the other player

Von Neumann's Maximin ("Best Worst Case") Decision Criterion

Von Neumann assumed that each player chooses a strategy in a position of "complete ignorance" concerning the other player's choice of strategy and, in fact, chooses 'safely'

Specifically, Ann chooses a strategy to solve

$$\max_{a_i} \min_{b_j} \pi^A(a_i, b_j)$$

Similarly, Bob chooses a strategy to choose

$$\max_{b_j} \min_{a_i} \pi^B(b_j, a_i)$$

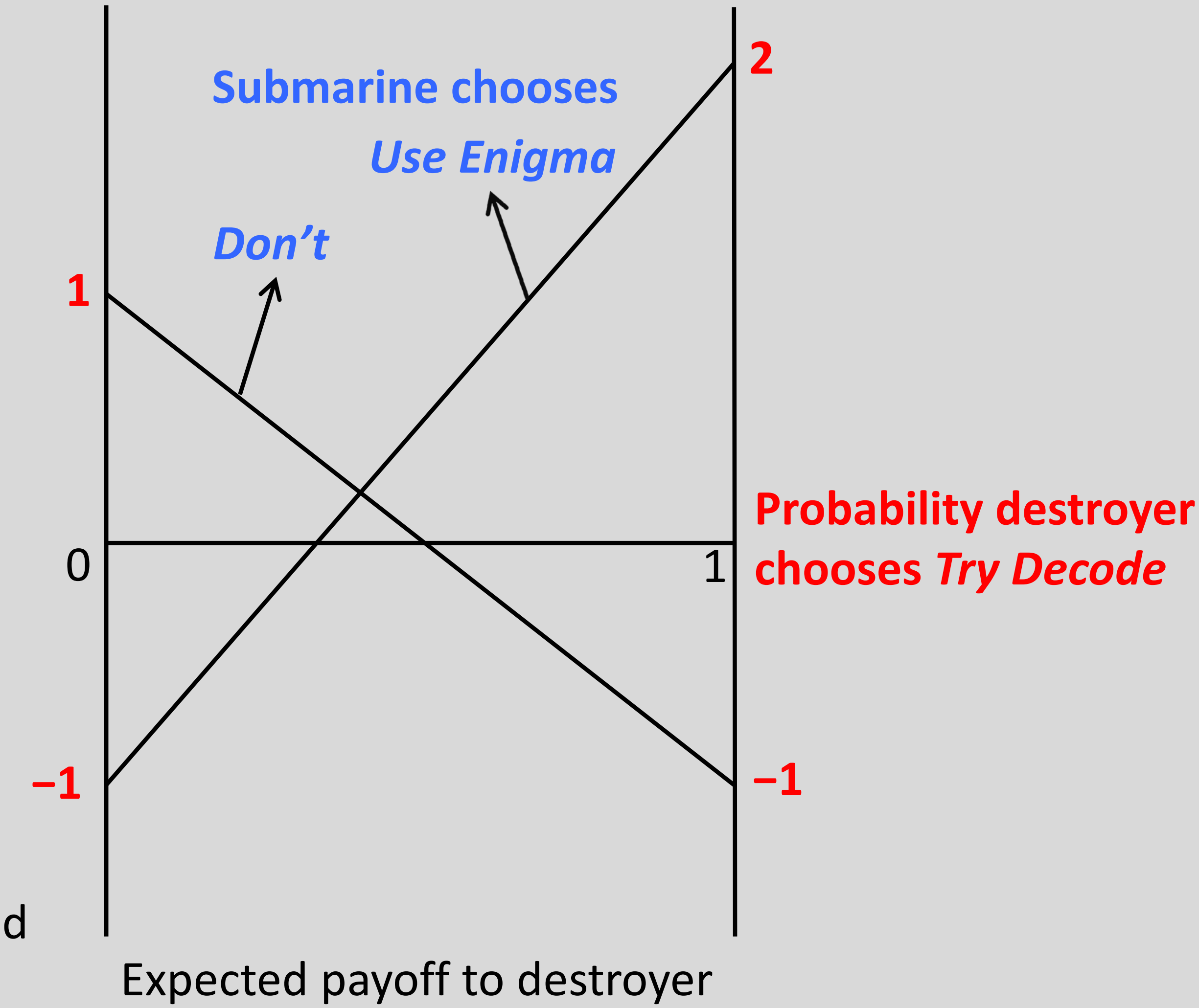
Notice that in von Neumann theory, players avoid a predictive approach (we will come back to this point)

In fact, von Neumann allowed for the possibility that players might deliberately choose strategies according to probability distributions they select (i.e., **mixed** vs. **pure** strategies)

Rationale for a Mixed Strategy

Destroyer \ Submarine	*Use Enigma*	*Don't*
Try Decode	2, −2	−1, 1
Don't	−1, 1	1, −1

"The Imitation Game" *



Note: We assume that the second player observes the first player's mixed strategy, not its realization

* See http://theimitationgamemovie.com; Winterbotham, F., *The Ultra Secret*, Harper & Row, 1974

Maximin Extended to Mixed Strategies

Let $p = (p_1, p_2, \dots, p_i, \dots)$ denote a mixed strategy for Ann, i.e., a probability distribution on the set $\{a_1, a_2, \dots, a_i, \dots\}$ of pure strategies for Ann

Let $q = (q_1, q_2, \dots, q_j, \dots)$ denote a mixed strategy for Bob, i.e., a probability distribution on the set $\{b_1, b_2, \dots, b_j, \dots\}$ of pure strategies for Bob

Ann now chooses a mixed strategy p to solve

$$\max_p \min_q \Pi^A(p, q)$$

where we now use the **expected payoff**

$$\Pi^A(p, q) = \sum_i \sum_j p_i \, q_j \pi^A(a_i, b_j)$$

We can write a similar expression for Bob

A Consistency Question

Ann is about to choose a maximin strategy

Before she does, she asks herself: “Suppose Bob is choosing a maximin strategy. What is my best choice of strategy in response to this?”

If the answer to this question is not a maximin strategy for Ann, then she might question the consistency of the maximin rule

The famous **Minimax Theorem**, due to von Neumann, implies that choice of a maximin strategy for Ann is optimal in response to choice of a maximin strategy by Bob

That is, the maximin decision criterion is consistent in this sense

Appendix: The Minimax Theorem in More Detail

Let p^* denote a maximin strategy for Ann, and q^* a maximin strategy for Bob

We want to prove that

$$\Pi^A(p^*, q^*) \geq \Pi^A(p, q^*) \forall p$$

$$\Pi^B(p^*, q^*) \geq \Pi^B(p^*, q) \forall q$$

The Minimax Theorem states that

$$\max_{a_i} \min_{b_j} \pi^A(a_i, b_j) = \min_{b_j} \max_{a_i} \pi^A(a_i, b_j)$$

(and likewise for Bob)

Exercise: Use the Minimax Theorem to prove the claim