Game Theory I:

Game Trees, Strategies, and Matrices

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n players $S_1, S_2, \ldots, S_n$ are playing a given game of strategy, $\mathcal{G}$. How must one of the participants, S_m, play in order to achieve a most advantageous result?

[I]t is inherent in the concept of "strategy" that all information about the actions of the participants and the outcomes of "draws" [i.e., moves by Nature] a player is able to obtain or infer is already incorporated in the "strategy." Consequently, each player must choose his strategy in complete ignorance of the choices of the rest of the players and of the results of the "draws."

What is the mathematical definition of a game?

What is the definition of a strategy?

How can one analyze choice under "complete ignorance"?

Von Neumann, J., "Zur Theorie der Gesellschaftsspiele," *Mathematische Annalen*, 100, 1928, 295- 320. (Bargman, S., English translation: "On the Theory of Games of Strategy," in Tucker, A., and R.D. Luce (eds.), *Contributions to the Theory of Games, Volume IV*, Princeton University Press, 1955. 13-42.)

A **game tree** is the proximate model of the situation being studied

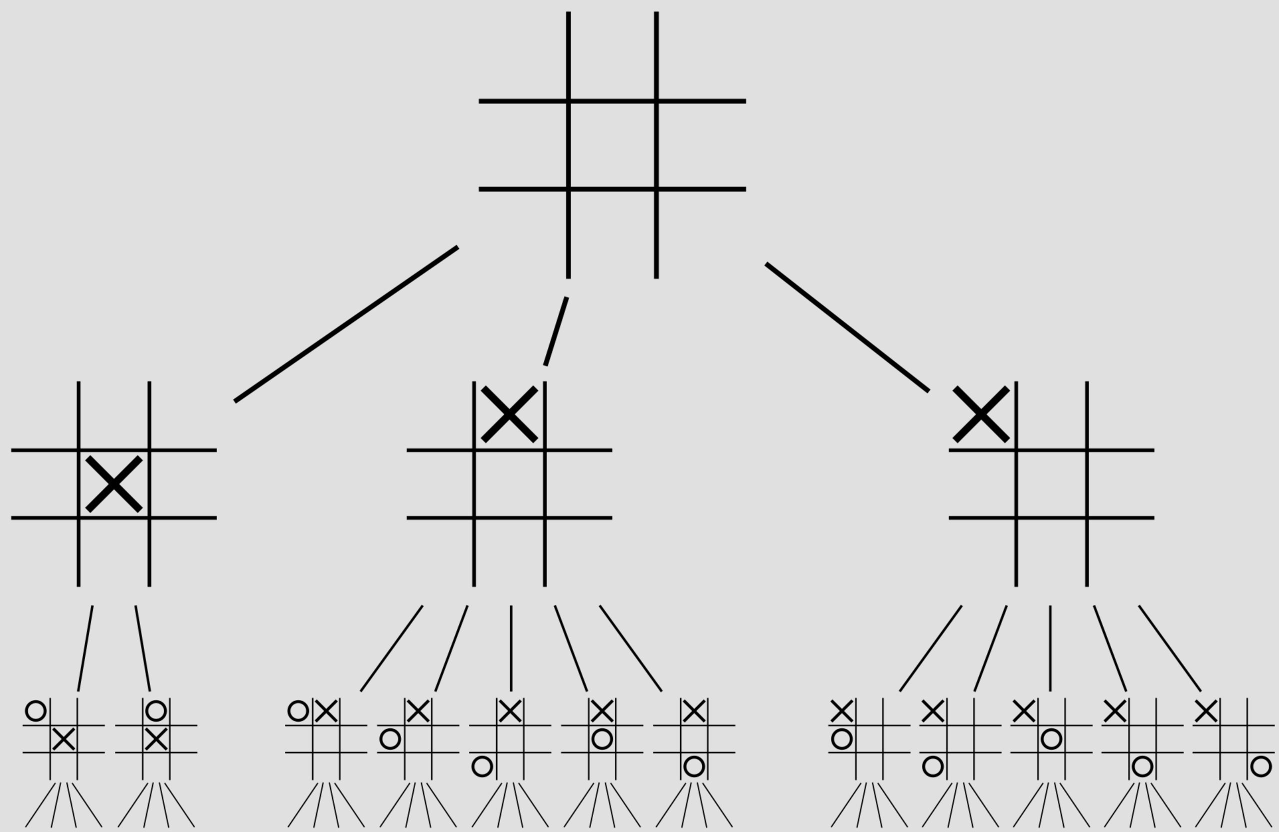
A game tree (reduced via symmetry) for tic-tac-toe:

t = 0:

t = 1:

t = 2:

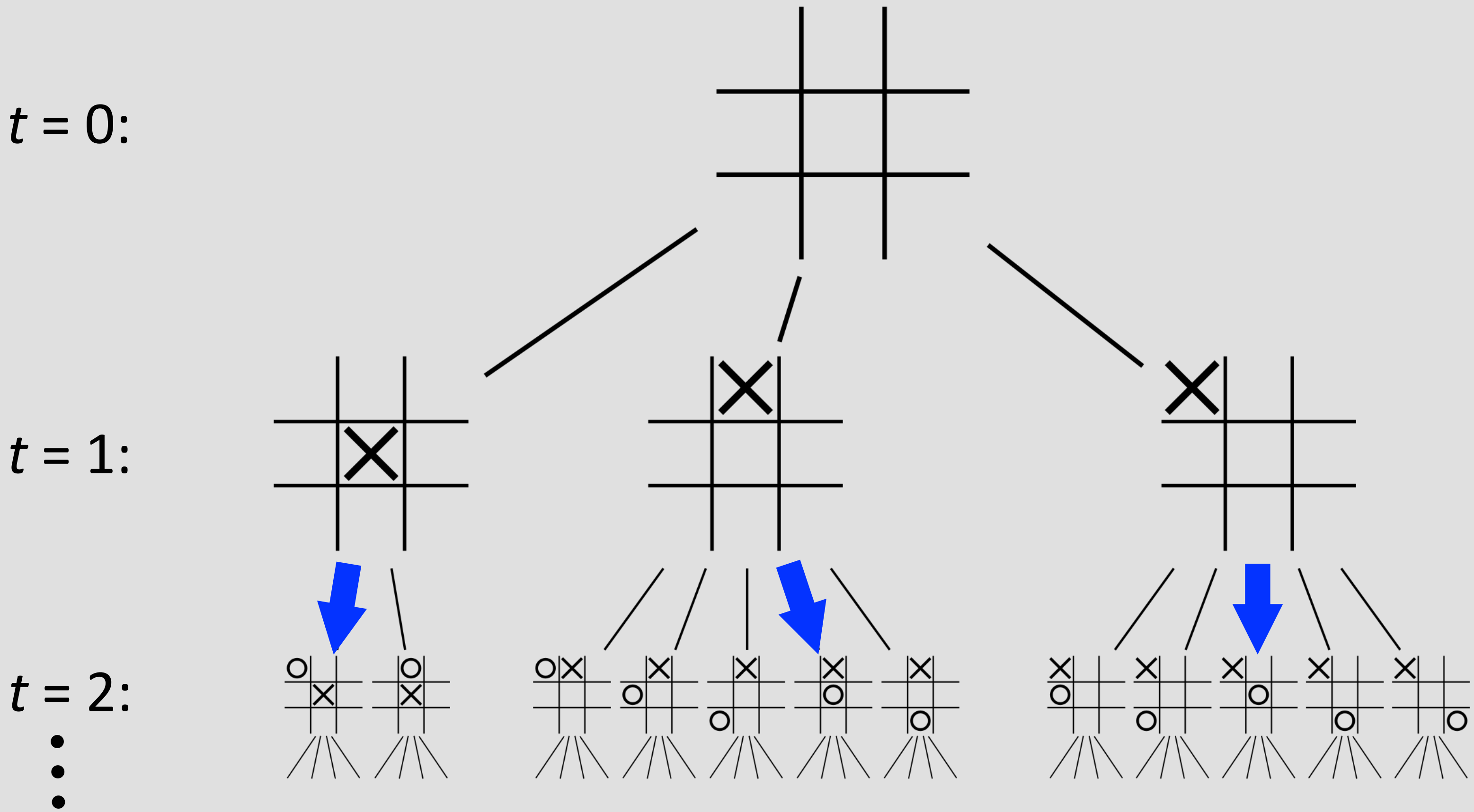
⋮



http://commons.wikimedia.org/wiki/File:Tic-tac-toe-game-tree.svg

A **strategy** for a player specifies, at each node (later: information set) that belongs to that player, a particular move

A partial strategy for the second mover in tic-tac-toe:



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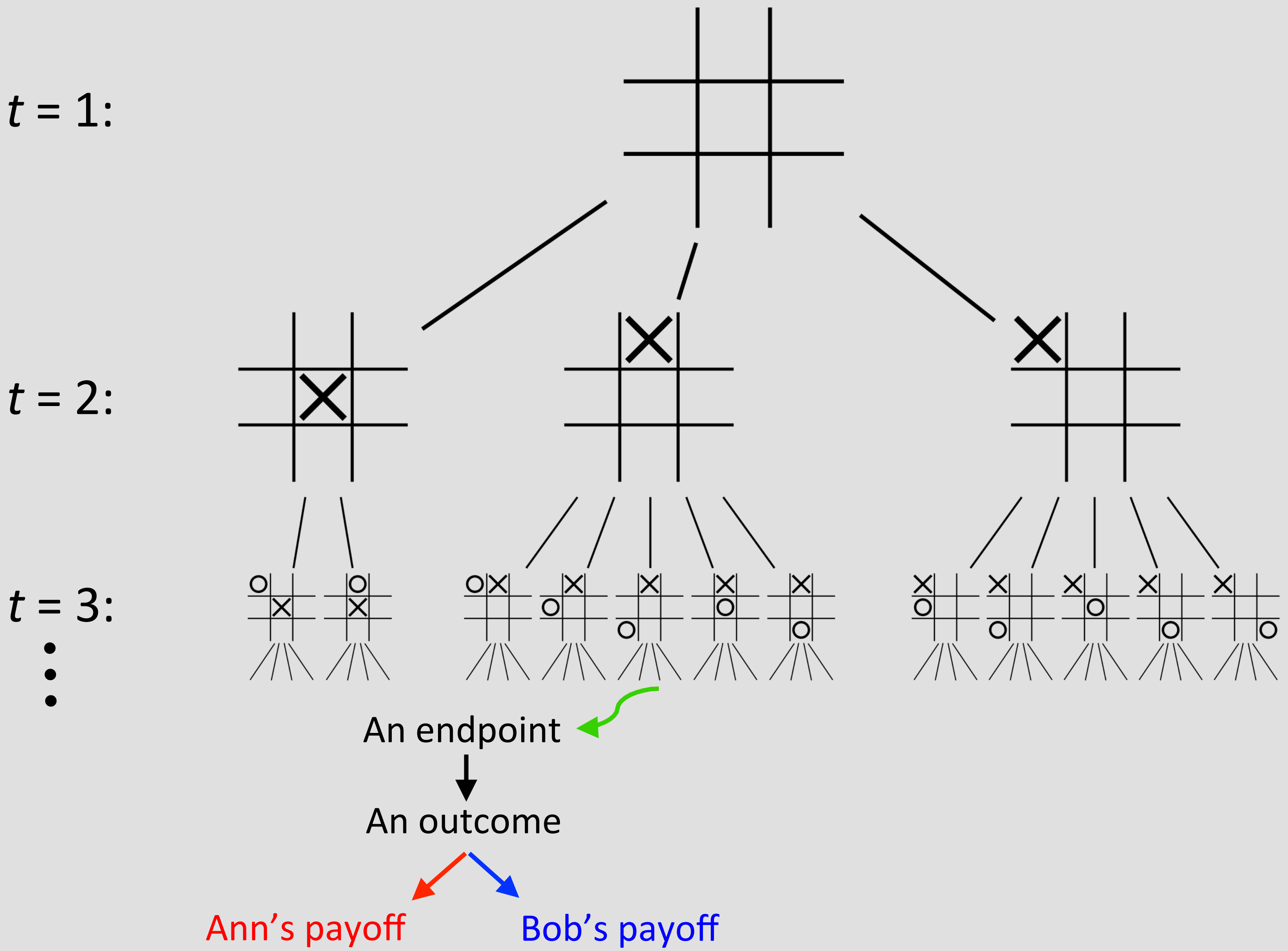


What if payoffs are not 'transparent' to the players? Or, even the tree? Later …

http://commons.wikimedia.org/wiki/File:Tic-tac-toe-game-tree.svg

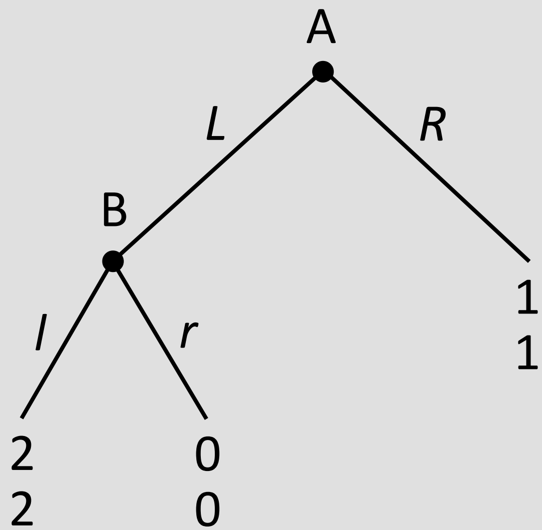
A **game matrix** is induced from a game tree

Bob's strategies

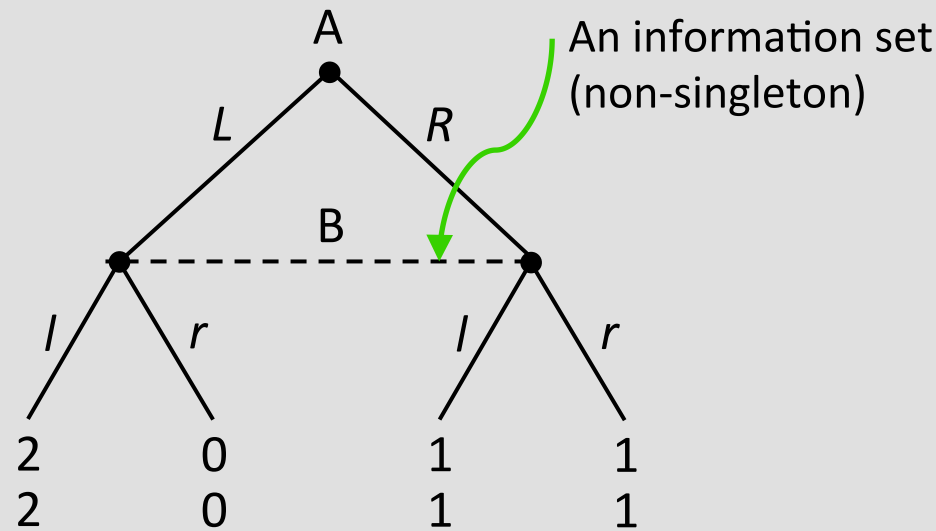
Ann's strategies

	b_1	•••	b_j	•••
a_1				
⋮				
a_i			$\pi^B(b_j, a_i)$ $\pi^A(a_i, b_j)$	
⋮				

The map from trees to matrices is many-to-one



(Perfect Information)



(Imperfect Information)

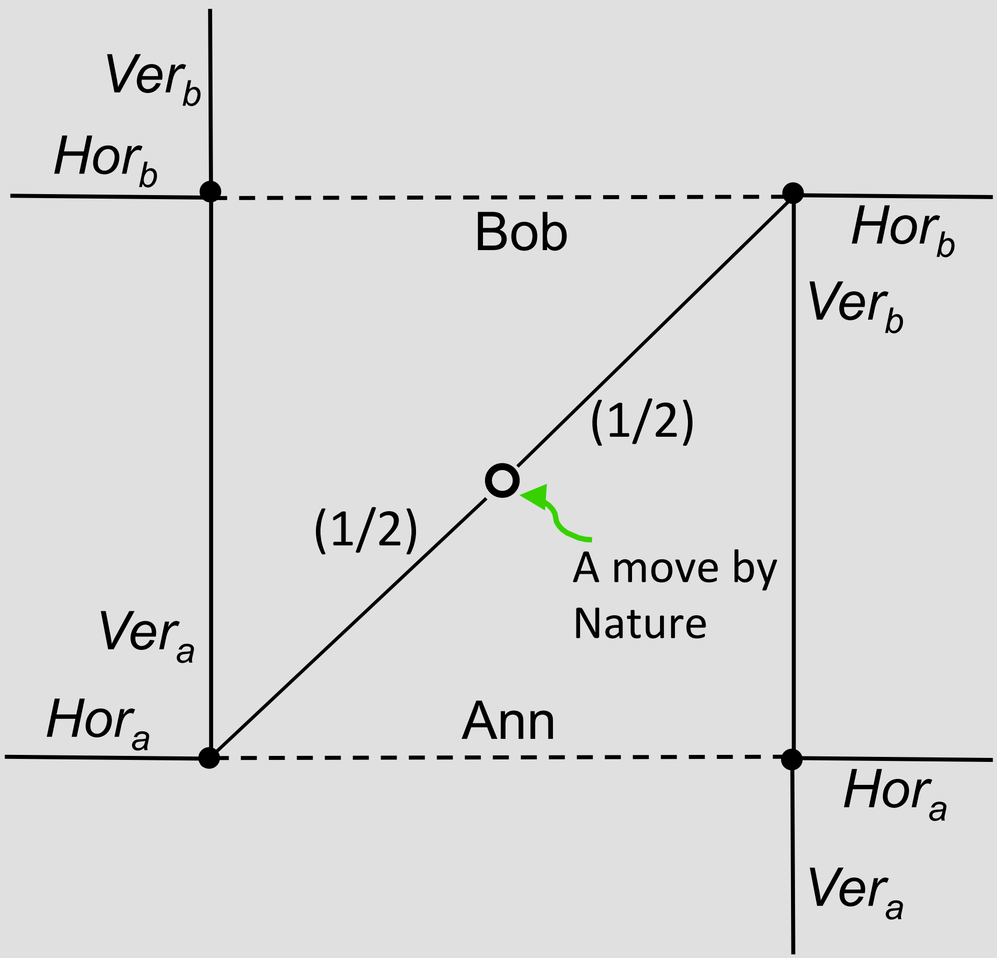
		B: *l*	B: *r*
A	*L*	2, 2	0, 0
A	*R*	1, 1	1, 1

Invariance can be investigated both analytically and behaviorally

Kohlberg, E., and J-F. Mertens, "On the Strategic Stability of Equilibria." *Econometrica*, 54, 1986, 1003-1037

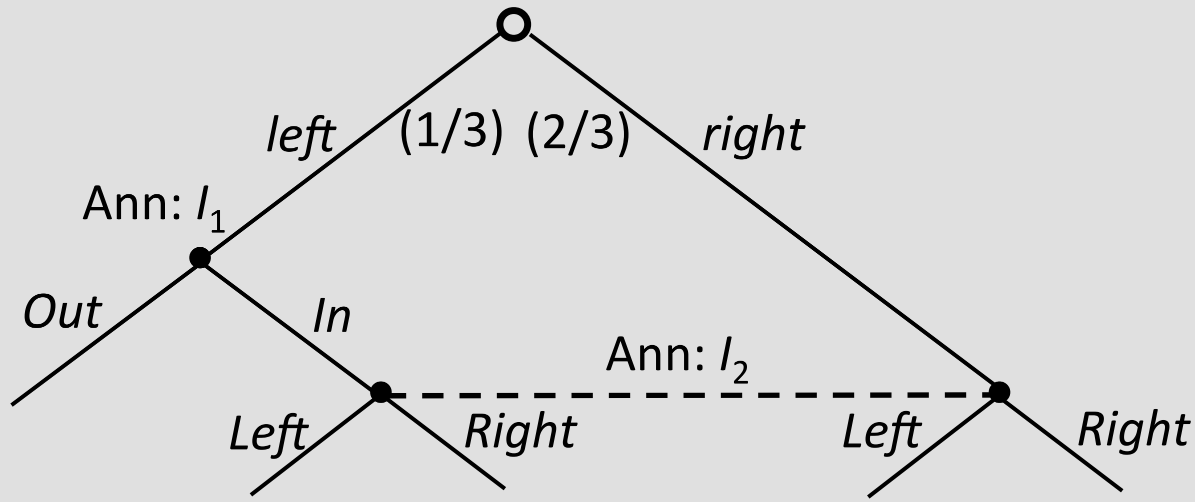
The **Kuhn tree** is a widely used formal definition

A Kuhn tree which is not a von Neumann tree ("no clock"):



 Kuhn, H., "Extensive Games and the Problem of Information," in Kuhn, H., and A. Tucker (eds.), *Contributions to the Theory of Games*, Vol. II, Annals of Mathematics Study 28, 1953, Princeton University Press, 193-216

A Kuhn tree with **imperfect recall**:



At information set I_2, Ann does not know if she moved previously (and chose *In*)

A **mixed strategy** for a player is a probability distribution on the player's (finite) set of strategies (the latter is then the set of **pure strategies** for the player)

A **behavioral strategy** for a player specifies, for each of the player's information sets, a probability distribution on the moves at that information set

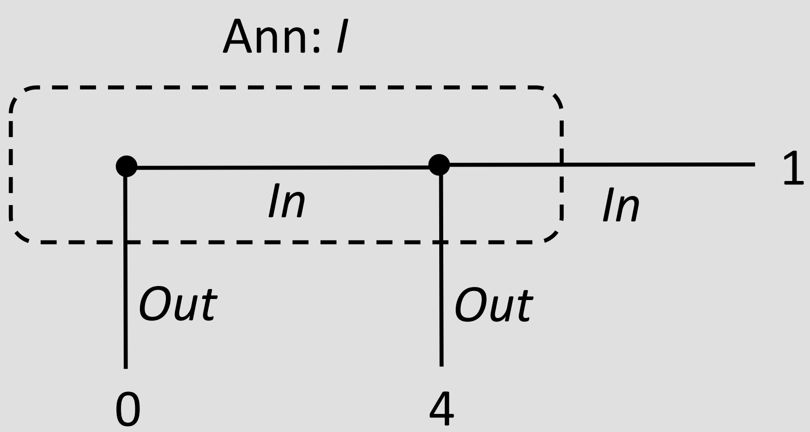
A mixed (resp. behavioral) strategy is a global (resp. local) concept

Kuhn's Theorem: Given a behavioral strategy we can build an equivalent* mixed strategy. If a player has perfect recall, then, given a mixed strategy for that player, we can build an equivalent* behavioral strategy

* For full definitions and proofs, see Kuhn op.cit.; also, Brandenburger, A., "A Note on Kuhn's Theorem," in van Benthem, J., D. Gabbay, and B. Loewe (eds.), *Texts in Logic and Games I: Interactive Logic: Proceedings of the 7th Augustus de Morgan Workshop*, Amsterdam University Press, 2007, 71-88

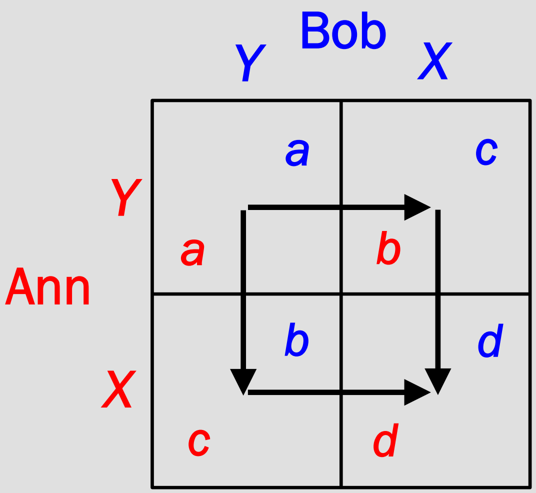
A **non-Kuhn** tree:

In a Kuhn tree, each path from the root to an endpoint crosses a given information set at most once

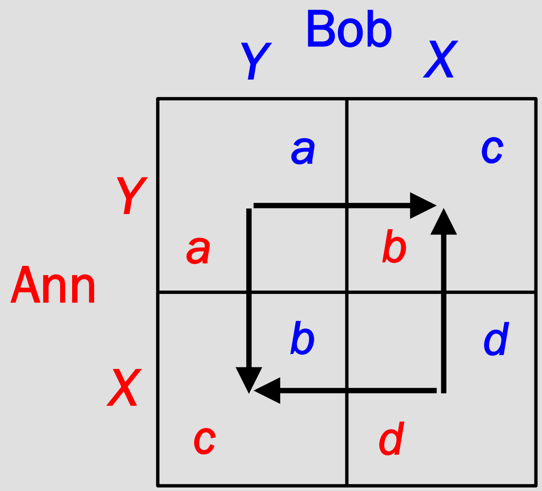


Isbell, J., "Finitary Games," in Drescher, M., A. Tucker, and P. Wolfe (eds.), *Contributions to the Theory of Games*, Vol. III, Annals of Mathematics Study 39, 1957, Princeton University Press, 79-96; Piccione, M., and A. Rubinstein, "On the Interpretation of Decision Problems with Imperfect Recall," *Games and Economic Behavior*, 20, 1997, 3-24

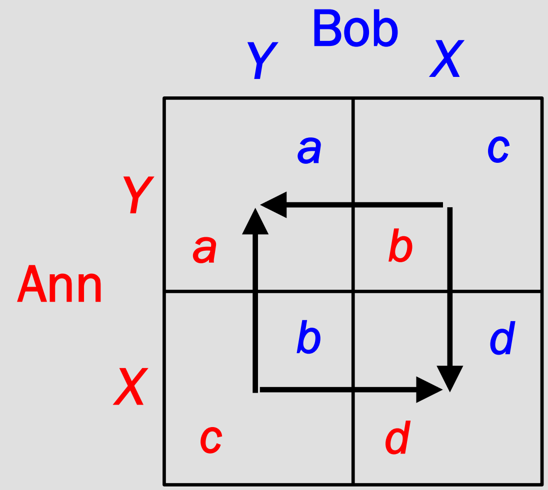
A classification of two-by-two game matrices:



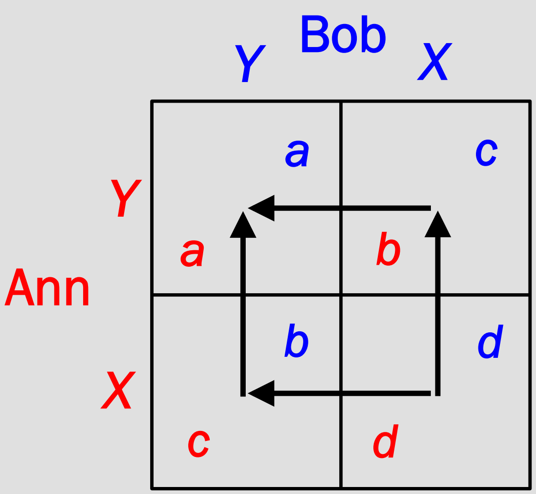
Prisoner's Dilemma



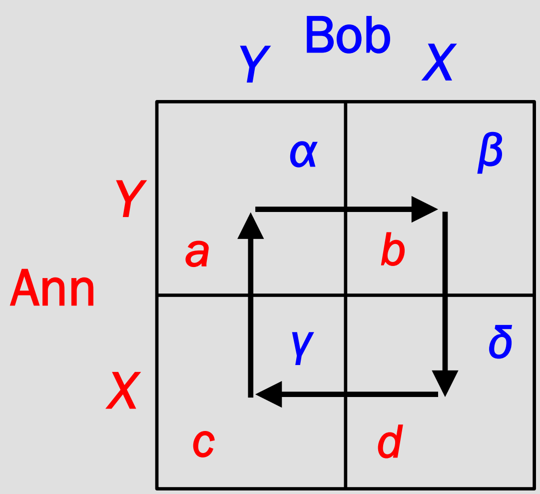
Battle of the Sexes



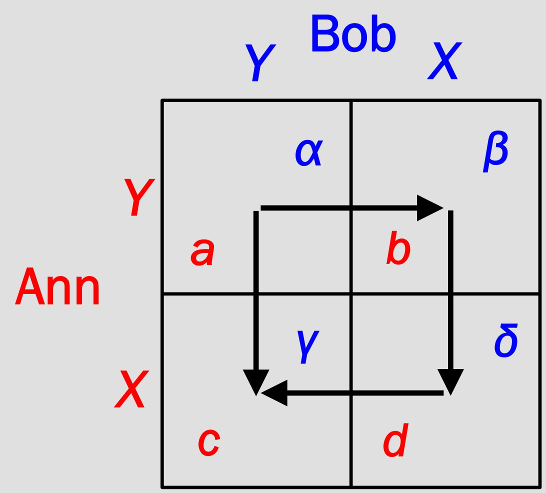
Coordination Game



Prisoner's Delight



Matching Pennies



One-Sided Game

 Assumptions: $a > d$ in the first four matrices (this is without loss of generality); $a \neq b \neq c \neq d$; $\alpha \neq \beta \neq \gamma \neq \delta$; thanks to Ken Corts