Notes on the Relationship Between Strong Belief and Assumption*

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In these notes, we define two maps, one from the set of conditional probability systems (CPS's) onto the set of lexicographic probability systems (LPS's), and another map from the set of LPS's with full support onto the set of CPS's. We use these maps to establish a relationship between strong belief (due to Battigalli-Siniscalchi [3, 2002] and defined on CPS's) and assumption (due to Brandenburger-Friedenberg-Keisler [5, 2004] and defined on LPS's). Related papers include Asheim-Søvik [1, 2005], Battigalli [2, 2004], Blume-Brandenburger-Dekel [4, 1989], Halpern [6, 2003], and Hammond [7, 1994].[1]

Our map from full-support LPS's to CPS's takes LPS's to the set of CPS's with a fixed set of open conditions. But to obtain a map from the set of CPS's onto the set of LPS's, we must allow the algebra of conditions to vary. When CPS's are used in game theory, there is usually a natural fixed family of conditions–corresponding to the events that a player can observe in the play of the game. These notes, then, are best viewed as an exercise in probability theory: we examine the relationship between CPS's and LPS's in a general, non-game setting.

1 Definitions

This section gives the main definitions considered throughout. Let Ω be a Polish space, and let $\mathcal{A}$ be the Borel σ-algebra on Ω. A set of **conditioning events** (or **conditions**) is a nonempty collection of sets in $\mathcal{A}$ that does not include the empty set. Throughout, we write $\mathcal{B}$ for a set of conditioning events.

Definition 1 *A **conditional probability system (CPS)** on $(\Omega, \mathcal{A}, \mathcal{B})$ is a map $p : \mathcal{A} \times \mathcal{B} \to [0,1]$ such that:*

a. for all $B \in \mathcal{B}$, $p(B|B) = 1$;

b. for all $B \in \mathcal{B}$, $p(\cdot|B)$ is a probability measure on $(\Omega, \mathcal{A})$;

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[1] A subsequent draft will contain a discussion of the literature.

c. for all $A \in \mathcal{A}$ and $B, C \in \mathcal{B}$, if $A \subseteq B \subseteq C$ then $p(A|C) = p(A|B)\, p(B|C)$.

Let $\mathcal{C}_{\mathcal{B}}$ be the set of all CPS's on $(\Omega, \mathcal{A}, \mathcal{B})$.

To map CPS's to LPS's, the set of conditions (associated with a CPS) will need to be sufficiently well-behaved. Here is the condition we will use:

Definition 2 *A set of conditioning events $\mathcal{B}$ will be called a* ***finite algebra*** *if $\mathcal{B} \cup \{\emptyset\}$ is a finite subalgebra of $\mathcal{A}$. A CPS on $(\Omega, \mathcal{A}, \mathcal{B})$ will be called a* ***finitary CPS*** *if $\mathcal{B}$ is a finite algebra. Write $\mathcal{C}$ for the set of all finitary CPS's.*

Next is the definition of strong belief, due to Battigalli-Siniscalchi [3, 2002]:

Definition 3 *Fix a CPS p. An event E is* ***strongly believed*** *under p if, for all $B \in \mathcal{B}$, $E \cap B \neq \emptyset$ implies $p(E|B) = 1$.*

Next is the definition of an LPS:

Definition 4 *A* ***lexicographic probability system (LPS)*** *on $(\Omega, \mathcal{A})$ is a finite sequence $\sigma = (\nu_0, \ldots, \nu_{n-1})$ of probability measures on $(\Omega, \mathcal{A})$ such that:*

a. for $i = 0, \ldots, n-1$, there are Borel sets U_i in Ω with $\nu_i(U_i) = 1$ and $\nu_i(U_j) = 0$ for $j \neq i$ (that is: σ is ***mutually singular****).*

Let $\mathcal{L}$ be the set of all LPS's on $(\Omega, \mathcal{A})$.

Note that, unlike CPS's, LPS's don't come with a fixed set of conditions. So, to map LPS's to CPS's we will need to specify a set of conditions. We will see that we can do this mapping so long as the LPS "covers" the set of conditions.

Definition 5 *An LPS $\sigma = (\nu_0, \ldots, \nu_{n-1})$ is said to* ***cover*** *$\mathcal{B}$ if, for each $B \in \mathcal{B}$ there is an $i < n$ with $\nu_i(B) > 0$. Let $\mathcal{L}_{\mathcal{B}}$ denote the set of all $\sigma \in \mathcal{L}$ such that σ covers $\mathcal{B}$.*

Next is the definition of assumption, due to Brandenburger-Friedenberg-Keisler [5, 2004]:

Definition 6 *Fix an LPS $\sigma = (\nu_0, \ldots, \nu_{n-1})$. An event E is* ***assumed (at level*** *j***)*** *under σ if $0 \leq j < n$ and:*

a. $\nu_i(E) = 1$ for all $i \leq j$;

b. $\nu_i(E) = 0$ for all $i > j$;

c. for each open set U with $E \cap U \neq \emptyset$, there is some i with $\mu_i(E \cap U) > 0$.

In [5, 2004], we show the following:

Lemma 1 *Fix an LPS $\sigma = (\nu_0, \ldots, \nu_{n-1})$. An event E is* ***assumed*** *(at level j) under σ if and only if σ satisfies conditions a-b of Definition 6 and:*

(a′) $E \subseteq \bigcup_{i=0}^{j} \operatorname{Supp} \nu_i$.

2 From CPS's to LPS's

In this section we will introduce a natural function f and show that it maps all finitary CPS's (i.e., across all $\mathcal{B}$ that are finite algebras) surjectively onto the LPS's. We show that if a finitary CPS p has full support and strongly believes E, then $f(p)$ has full support and assumes E. We will then give a characterization of assumption: Fix a full-support LPS σ. An event E is assumed under σ if and only if there is a finite algebra $\mathcal{B}$ and a full-support CPS $p \in f^{-1}(\sigma)$ on $(\Omega, \mathcal{A}, \mathcal{B})$ where E is strongly believed under p.

2.1 The Natural Map

We will first show that given any CPS $p \in \mathcal{C}$, there is a naturally induced LPS $f(p)$ in $\mathcal{L}$. We use the following fact:

F1 A finite subalgebra $\mathcal{B}$ of $\mathcal{A}$ is generated by a finite partition. We refer to the cells of the partition as **atoms** of $\mathcal{B}$.

Fix a finite algebra of conditions $\mathcal{B}$. Adopt the convention that $V_{-1} = \emptyset$. For a given $p \in \mathcal{C}_{\mathcal{B}}$, we will inductively define μ_i, V_i, and W_i as follows: Let $W_0 = \Omega \backslash V_{-1}$ and $\mu_0 = p(\cdot|W_0)$. Take V_0 to be the union over all atoms $B \in \mathcal{B}$ with $\mu_0(B) > 0$. Assume μ_i, V_i, W_i are defined. If $\Omega = \bigcup_{k \le i} V_k$, then define $\sigma = (\mu_0, \ldots, \mu_i)$ and put $n = i+1$. Otherwise, let $W_{i+1} = \Omega \backslash \bigcup_{k \le i} V_k$ and $\mu_{i+1} = p(\cdot|W_{i+1})$. Let V_{i+1} be the union over atoms $B \in \mathcal{B}$ with $\mu_{i+1}(B) > 0$. Note that these sets are well defined since, for each i, $V_i, W_i \in \mathcal{B}$. (This uses the fact that $\mathcal{B}$ is a finite algebra.) Also, notice that the construction terminates at some finite $n = i+1$, since $\mathcal{B}$ is finite. We note the following additional facts:

F2 $V_i \subseteq W_i = \Omega \backslash \bigcup_{k<i} V_k$.

F3 $W_{i+1} \subseteq W_i$.

Definition 7 *For each CPS $p \in \mathcal{C}$, $f(p)$ is the sequence $f(p) = (\mu_0, \ldots, \mu_{n-1})$.*

Lemma 2 *For each $i < n$, $p(\cdot \mid V_i) = p(\cdot \mid W_i) = \mu_i$.*

Proof. Fix a set $A \in \mathcal{A}$. Since $p(\cdot \mid W_i)$ is a probability measure, we have $p(V_i \mid W_i) = 1$. By F3 and the definition of a CPS, $p(A \mid W_i) = p(A \mid V_i)p(V_i \mid W_i) = p(A \mid V_i)$. ■

Lemma 3 *The sets V_i, $i = 0, \ldots, n-1$, form a partition of Ω with*

(i) $\mu_i(V_i) = 1$ for all $i = 0, \ldots, n-1$; and

(ii) $\mu_i(V_j) = 0$ for all $j \neq i$.

Proof. By construction $\Omega = \bigcup_{i<n} V_i$, and V_i and V_j are disjoint whenever $i \neq j$ (by F2). By Lemma 2, $\mu_i(V_i) = 1$ for each i. For $j < i$, $W_i \subseteq \Omega \backslash V_j$ (by F3), so that $\mu_i(V_j) = p(V_j|W_i) = 0$. For $j > i$, $V_j \subseteq W_j = \Omega \backslash \bigcup_{k<j} V_k$. In particular, $V_j \subseteq \Omega \backslash V_i$. Since $\mu_i(V_i) = 1$, $\mu_i(\Omega \backslash V_i) = \mu_i(V_j) = 0$. ■

Lemma 4 *For each CPS $p \in \mathcal{C}$, $f(p)$ is an LPS on $(\Omega, \mathcal{A})$. That is, f maps $\mathcal{C}$ into $\mathcal{L}$.*

Proof. Mutual singularity follows immediately from Lemma 3. ■

Proposition 1 *The map f takes $\mathcal{C}$ surjectively onto $\mathcal{L}$.*

Proof. Let $\sigma = (\nu_0, \ldots, \nu_{n-1}) \in \mathcal{L}$. By definition there are Borel sets U_i such that $\nu_i(U_i) = 1$ and $\nu_j(U_i) = 0$ for $i \neq j$. Let $X_0 = U_0$, for $0 < i < n-1$ let $X_i = U_i \setminus \bigcup_{k<i} X_k$, and let $X_{n-1} = \Omega \setminus \bigcup_{k<n-1} X_k$. The sets $X_0, \ldots, X_{n-1}$ form a partition of Ω into Borel sets such that $\nu_i(X_i) = 1$ and $\nu_j(X_i) = 0$ for $j \neq i$. Let $\mathcal{B}$ be the unique algebra with the atoms $X_0, \ldots, X_{n-1}$.

We define a mapping $p : \mathcal{A} \times \mathcal{B} \to [0,1]$ and show that $p \in \mathcal{C}_{\mathcal{B}}$ and $f(p) = \sigma$. For each $B \in \mathcal{B}$, there is a least $i < n$ such that $X_i \subseteq B$. For each $Y \in \mathcal{A}$, define $p(Y \mid B) = \nu_i(Y)$.

We now show that p is a CPS on $(\Omega, \mathcal{A}, \mathcal{B})$. Note, conditions a–b are immediate. To prove c, fix $A \in \mathcal{A}$ and $B, C \in \mathcal{B}$ with $A \subseteq B \subseteq C$. Take the least i and j such that $X_i \subseteq B$ and $X_j \subseteq C$. Since $B \subseteq C$, $i \geq j$. If $i = j$, then $p(B \,|\, C) = \nu_j(B) = \nu_i(B) = 1$, so

$$p(A \,|\, C) = \nu_j(A) = \nu_i(A) \cdot 1 = p(A \,|\, B) p(B \,|\, C),$$

and c holds. If $i > j$, we have $p(B \,|\, C) = \nu_j(B) = 0$, so

$$p(A \,|\, C) = \nu_j(A) = 0 = p(A \,|\, B) p(B \,|\, C),$$

and c holds in this case as well. Thus $p \in \mathcal{C}_{\mathcal{B}}$.

Finally, starting with p, we have by induction that $V_i = X_i$ for each $i < n$. By Lemma 2, $\mu_i = p(\cdot \,|\, W_i) = p(\cdot \,|\, V_i) = p(\cdot \,|\, X_i) = \nu_i$, and thus $f(p) = \sigma$. ■

The above proof shows the following additional fact.

Corollary 1

(i) For each finitary CPS $p \in \mathcal{C}_{\mathcal{B}}$, the length of $f(p)$ is at most the number of atoms of $\mathcal{B}$.

(ii) For each LPS $\sigma \in \mathcal{L}$ there exists a finite algebra of conditions $\mathcal{B}$ and a CPS $p \in \mathcal{C}_{\mathcal{B}}$ such that $f(p) = \sigma$ and the number of atoms of $\mathcal{B}$ is equal to the length of σ.

Remark 1 *Suppose Ω is infinite and $\mathcal{B}$ is a finite algebra. There are LPS's of all finite lengths, and therefore f never maps $\mathcal{C}_{\mathcal{B}}$ surjectively onto $\mathcal{L}$.*

Example 1 *The purpose of this example is to show that f is not injective from $\mathcal{C}$ into $\mathcal{L}$. Let $\Omega = \{a, b\}$, $\mu(\{a\}) = \mu(\{b\}) = 1/2$, and let $\mathcal{A}$ be the power set of Ω, let $\mathcal{B} = \{\Omega, \{a\}, \{b\}\}$, and let $\mathcal{D}$ be the trivial algebra $\{\Omega\}$. Let p be the CPS on $(\Omega, \mathcal{A}, \mathcal{B})$ and q be the CPS on $(\Omega, \mathcal{A}, \mathcal{D})$ obtained by conditioning μ. Then $f(p) = f(q) = (\mu)$, but p and q are distinct because their condition algebras $\mathcal{B}$ and $\mathcal{D}$ are different.*

Now we turn to the relationship between strong belief and assumption, under the mapping f.

2.2 Full Support

This section establishes a relationship between full-support CPS's and full-support LPS's.

Definition 8 *A CPS p has **full support** if, for each atom $B \in \mathcal{B}$, $B \subseteq \operatorname{Supp} p\,(\cdot|B)$.*

Definition 9 *An LPS $\sigma = (\nu_0, \ldots, \nu_{n-1})$ has **full support** if $\Omega = \bigcup_{i<n} \operatorname{Supp} \nu_i$.*

Lemma 5 *Fix a CPS p and events $B, C \in \mathcal{B}$ with $B \subseteq C$ and $p\,(B|C) > 0$. Then $\operatorname{Supp} p\,(\cdot|B) \subseteq \operatorname{Supp} p\,(\cdot|C)$.*

Proof. Let $A = \operatorname{Supp} p(\cdot \mid C)$. We have $p(A \mid C) = 1$ and $p(B \mid C) > 0$, so $p(A \cap B \mid C) = p(B \mid C)$. Since p is a CPS,

$$p(A \cap B \mid C) = p(A \cap B \mid B) p(B \mid C).$$

Therefore $p(A \mid B) \geq p(A \cap B \mid B) = 1$. Since A is also closed, it must contain $\operatorname{Supp} p(\cdot \mid B)$. ■

Lemma 6 *Fix a full-support finitary CPS p on $(\Omega, \mathcal{A}, \mathcal{B})$ and the associated LPS $f(p) = (\mu_0, \ldots, \mu_{n-1})$. Then $V_i \subseteq \operatorname{Supp} \mu_i$ for each $i < n$.*

Proof. Each of the sets V_i can be written as the union of atoms $B \in \mathcal{B}$ where $p(B|W_i) > 0$, or equivalently, using Lemma 2, where $p(B|V_i) > 0$. So Lemma 5 implies $\operatorname{Supp} p(\cdot|B) \subseteq \operatorname{Supp} p(\cdot|V_i)$. But $B \subseteq \operatorname{Supp} p(\cdot|B)$, since p has full support. So $B \subseteq \operatorname{Supp} p(\cdot|V_i)$. Finally, $\operatorname{Supp} p(\cdot|V_i) = \operatorname{Supp} \mu_i$, by Lemma 2 again. ■

Proposition 2 *The map f takes the set of full-support finitary CPS's surjectively onto the set of full-support LPS's. That is, an LPS σ has full support if and only if there exists some finite algebra $\mathcal{B}$ and some full-support CPS p on $(\Omega, \mathcal{A}, \mathcal{B})$ with $\sigma = f(p)$.*

Proof. Fix a full-support CPS p and let the associated LPS be $f(p) = (\mu_0, \ldots, \mu_{n-1})$. By Lemmas 3 and 6,

$$\Omega = \bigcup_{i \leq n-1} V_i \subseteq \bigcup_{i \leq n-1} \operatorname{Supp} \mu_i,$$

as required.

Now let $\sigma = (\nu_0, \ldots, \nu_{n-1})$ be a full-support LPS. In the proof of Proposition 1, the sets X_i may be taken so that $X_i \subseteq \operatorname{Supp} \nu_i$ for each $i < n$. This can be done by first replacing each of the original sets X_i by its intersection with $\operatorname{Supp} \nu_i$. Then put each point ω of $\Omega \setminus \bigcup_{i<n} X_i$ into X_j where j is the first integer such that $\omega \in \operatorname{Supp} \nu_j$. The proof of Proposition 1 will now give us a full-support CPS p such that $f(p) = \sigma$. ■

Example 2 *This is an example of a finitary CPS p such that $f(p)$ has full support but p does not. Let Ω be the real interval $[0, 2]$, and let $\mathcal{A}$ be the algebra of Borel subsets of Ω. Pick a point $a \in (1, 2)$ and let $Y = [0, 1] \cup \{a\}$ and $\mathcal{B} = \{[0, 2], Y, [0, 2] \setminus Y\}$. Set $p(\cdot|\Omega)$ to be the uniform Borel measure conditioned on $[0, 1]$ and set $p(\cdot|\, [0, 2] \setminus Y)$ to be the uniform Borel measure conditioned on $[1, 2]$. Then $f(p) = (\mu_0, \mu_1)$ is such that $\mu_0 = p(\cdot|\Omega)$ and $\mu_1 = p(\cdot|\, [0, 2] \setminus Y)$. The LPS $f(p)$ has full support, but p does not have full support because $a \in Y \setminus \operatorname{Supp} p(\cdot \mid Y)$.*

2.3 From Strong Belief to Assumption

Proposition 3 *Fix a full-support finitary CPS p. If E is strongly believed under p then E is assumed under $f(p)$.*

Proof. Suppose E is strongly believed under p. Then $\mu_0(E) = p(E|\Omega) = 1$. Also note that, by F3, if $E \cap W_i = \emptyset$ then $E \cap W_{i+1} = \emptyset$. So, there is $j \geq 0$ with: (i) $\mu_i(E) = 1$ for all $i \leq j$, and (ii) $E \cap W_i = \emptyset$ and $\mu_i(E) = 0$ for all $i > j$. Now note that $E \cap W_{j+1} = \emptyset$ implies $E \subseteq \bigcup_{i \leq j} V_i$. By Lemma 6, $E \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$ as required. ■

Example 3 *The purpose of this example is to show that, if a finitary CPS p does not have full support, then an event E can be strongly believed under p even though it is not assumed under $f(p)$.*

Let $\Omega = \{a, b, c, d\}$ and $\mathcal{B} = \{\Omega, \{a, b\}, \{c\}\}$. Let $p(a|\Omega) = p(a|\{a, b\}) = 1$ and $p(c|\{c\}) = 1$. Then $f(p) = (\mu_0, \mu_1)$ where $\mu_0(a) = 1$ and $\mu_1(c) = 1$. Take $E = \{a, b\}$ and note that p strongly believes E but $f(p)$ does not assume E. (Condition c is violated.)

Lemma 7 *Suppose $\sigma = (\nu_0, \ldots, \nu_{n-1})$ is an LPS with full support and the set E is assumed by σ at level j. Then there is a partition of Ω into Borel sets $U_0, \ldots, U_{n-1}$ such that $E \subseteq U_0 \cup \cdots \cup U_j$ and for each $i < n$, $\nu_i(U_i) = 1$ and $U_i \subseteq \operatorname{Supp} \nu_i$.*

Proof. We argue by induction on j. Suppose E is assumed by σ at level 0. By the proof of Proposition 2, there is a partition $X_0, \ldots, X_{n-1}$ of Ω such that for each $i < n$, $\nu_i(X_i) = 1$ and $X_i \subseteq \operatorname{Supp} \nu_i$. Take $U_0 = E \cup X_0$, and $U_i = X_i \setminus E$ for $0 < i < n$. Since E is assumed at level 0, certainly $\nu_0 (U_0) = 1$ and $U_0 \subseteq \operatorname{Supp} \nu_0$. For $i \geq 1$, note that, using the fact that E is assumed, $\nu_i (X_i \cap E) = 0$, so that $\nu_i (U_i) = 1$. Moreover, $X_i | E \subseteq X_i \subseteq \operatorname{Supp} \nu_i$, as required.

Now suppose E is assumed by σ at level $j \geq 1$. Choose a set F which is assumed by σ at level $j - 1$. Let $G = (E \setminus F) \cap \operatorname{Supp} \nu_j$. Then $\nu_j (F) = 0$ so that $\nu_j (G) = 1$. Certainly, $G \subseteq \operatorname{Supp} \nu_j$. Also note that, for $k \neq j$, $\nu_k (G) = 0$. For $k < j$, this follows from the fact that $\nu_k (F) = 0$ so that $\nu_k (E \backslash F) = 0$. When $k > j$, this follows from the fact that E is assumed at level j.

Using these facts, the set $E \backslash G$ is assumed by σ at level $j-1$. To see this note that $j-1 \geq 0$. For all $k \leq j - 1$, $\nu_k (E) = 1$ and $\nu_k (G) = 0$, since $j \neq k$, so that $\nu_k (E \backslash G) = 1$. Moreover, $\nu_j (G) = 1$ so that $\nu_j (E \backslash G) = 0$. For all $k > j$, $\nu_k (E) = 0$ so that $\nu_k (E \backslash G) = 0$. Finally, by assumption, $E \backslash G \subseteq \bigcup_{k \leq j} \operatorname{Supp} \nu_k$. By construction $(E \backslash G) \cap \operatorname{Supp} \nu_j = \emptyset$. So, $E \backslash G \subseteq \bigcup_{k \leq j-1} \operatorname{Supp} \nu_k$.

The inductive hypothesis for $E \backslash G$ gives us a partition of Ω into Borel sets $Y_0, \ldots, Y_{n-1}$ such that $E \backslash G \subseteq Y_0 \cup \cdots \cup Y_{j-1}$, and $\nu_i(Y_i) = 1$ and $Y_i \subseteq \operatorname{Supp} \nu_i$ for each $i < n$. The required sets are $U_i = Y_i$ for $i < j$, $U_j = Y_j \cup G$, and $U_i = Y_i \backslash G$ for $i > j$. ■

Proposition 4 *Let $\sigma = (\nu_0, \ldots, \nu_{n-1})$ be an LPS with full support. A set E is assumed under σ if and only if E is strongly believed under some finitary CPS p with full support such that $f(p) = \sigma$.*

Proof. One direction follows from Proposition 3.

For the other direction, suppose that E is assumed under σ at level j. Let $U_0, \ldots, U_{n-1}$ be the partition given by Lemma 7. Let $\mathcal{B}$ be the finite algebra of sets with atoms $U_0, \ldots, U_{n-1}$. The proof of Proposition 2 gives us a CPS $p \in \mathcal{C}_{\mathcal{B}}$ with full support such that $f(p) = \sigma$ and for each i, $\nu_i = p(\cdot \,|\, W_i) = p(\cdot \,|\, U_i)$ (where the last equality follows from Lemma 2). For each $i \leq j$ we have $v_i (E) = p(E \,|\, W_i) = 1$, and for each $i > j$ we have $E \cap U_i = \emptyset$.

Fix an event $B \in \mathcal{B}$ and note that B can be taken to be a union of atoms. Suppose $E \cap B \neq \emptyset$. Then there exists some $U_i \subseteq B$ with $E \cap U_i \neq \emptyset$. It follows $i \leq j$ and $p (E | U_i) = 1$. Choose U_i so that there does not exist some $U_k \subseteq B$ with $E \cap U_k \neq \emptyset$ and $k < i$. Then $B \subseteq W_i = \Omega \backslash \bigcup_{k<i} U_k$. Using condition c of a CPS,

$$p (E \cap B | W_i) = p (E \cap B | B) \, p (B | W_i) .$$

We have that $p (U_i | W_i) = \nu_i (U_i) = 1$ and so $p (B | W_i) = 1$. Also,

$$p (E \cap B | W_i) = p (E \cap B | U_i) = p (E | U_i) = 1,$$

where the first equality follows from Lemma 2 and the second follows from the fact that $U_i \subseteq B$. But then,

$$1 = p (E \cap B | B) \times 1,$$

or $p (E | B) = 1$. ■

Note that in the above proof, the algebra $\mathcal{B}$ is not fixed in advance, and depends on σ and E.

3 From LPS's to CPS's

For each set of conditions $\mathcal{B}$, we will introduce a function $g_{\mathcal{B}}$ from LPS's to CPS's. This function will map $\mathcal{L}_{\mathcal{B}}$ surjectively onto the set $\mathcal{C}_{\mathcal{B}}$. An important case in this section will be when each conditioning event in $\mathcal{B}$ is open. We will show that, in this case, if an event E is assumed under some $\sigma \in L_{\mathcal{B}}$, then E is strongly believed under $g_{\mathcal{B}}(\sigma)$. We go on to provide a number of characterizations of strong belief. Notably: When $\mathcal{B}$ is an open finite algebra, a CPS $p \in C_{\mathcal{B}}$ strongly believes E if and only if E is assumed by some LPS $\sigma \in g_{\mathcal{B}}^{-1}(p)$ with full support.

3.1 The Natural Map

Lemma 8 *Fix an LPS $\sigma = (\nu_0, \ldots, \nu_{n-1}) \in \mathcal{L}_{\mathcal{B}}$. Let $p : \mathcal{A} \times \mathcal{B} \to [0,1]$ be the function such that, for each $B \in \mathcal{B}$, $p(\cdot|B) = \nu_k(\cdot|B)$ where k is the least integer such that $\nu_k(B) > 0$. Then p is a CPS.*

Proof. Conditions a–b are immediate. Fix $A \in \mathcal{A}$, $B, C \in \mathcal{B}$ with $A \subseteq B \subseteq C$. Let j be the minimum i with $\nu_i(B) > 0$ and let k be the minimum i with $\nu_i(C) > 0$. (This is well-defined since σ covers $\mathcal{B}$.) Note $k \le j$ since $B \subseteq C$. First suppose that $\nu_k(B) > 0$. Then using the fact that $A \subseteq B \subseteq C$,

$$p(A|C) = \frac{\nu_k(A)}{\nu_k(C)} = p(A|B)\, p(B|C).$$

Next suppose that $\nu_k(B) = 0$. Then $\nu_k(A) = 0$ so that $p(A|C) = 0 = p(A|B)\, p(B|C)$. ■

Definition 10 *Fix a set of conditioning event $\mathcal{B}$ and let $g_{\mathcal{B}} : \mathcal{L}_{\mathcal{B}} \to \mathcal{C}_{\mathcal{B}}$ map each $\sigma \in \mathcal{L}_{\mathcal{B}}$ into the CPS defined by Lemma 8.*

We first relate the map f to the space $\mathcal{L}_{\mathcal{B}}$ and the map $g_{\mathcal{B}}$.

Example 4 *This example shows that $\mathcal{L}_{\mathcal{B}}$ can be a proper subset of $\mathcal{L}$. Let $\Omega = \{a,b\}$ and $\mathcal{B} = \{\Omega, \{a\}, \{b\}\}$. Let $\sigma = (\mu_0)$ where $\mu_0(a) = 1$. Then $\sigma \notin \mathcal{L}_{\mathcal{B}}$ because no μ_i assigns positive measure to the atom $\{b\}$. Note, however, there exists a subalgebra $\mathcal{D} = \{\Omega\}$ and p is the CPS in $\mathcal{C}_{\mathcal{D}}$, given by $p(\cdot|\Omega) = \mu_0$, so that $\sigma = f(p)$.*

Example 5 *The purpose of this example is to show that in general f does not map $\mathcal{C}_{\mathcal{B}}$ onto $\mathcal{L}_{\mathcal{B}}$. Let $\Omega = \{a,b,c,d\}$ and $\mathcal{B}$ be the subalgebra with atoms $\{a,b\}$ and $\{c,d\}$. Let $\sigma = (\mu_0, \mu_1)$ be the full-support LPS with $\mu_0(a) = \mu_0(c) = 1/2$ and $\mu_1(b) = \mu_1(d) = 1/2$. Then there is no $p \in \mathcal{C}_{\mathcal{B}}$ such that $f(p) = \sigma$ because every μ_i assigns positive measure to both atoms of $\mathcal{B}$. However, taking $\mathcal{D}$ to be the subalgebra with atoms $\{a,c\}$ and $\{b,d\}$, one can easily find a CPS $p \in \mathcal{C}_{\mathcal{D}}$ with $f(p) = \sigma$.*

Proposition 5 *Fix a finite algebra of conditions $\mathcal{B}$.*

(i) The restriction of f to $\mathcal{C}_{\mathcal{B}}$ maps $\mathcal{C}_{\mathcal{B}}$ injectively into $\mathcal{L}_{\mathcal{B}}$.

(ii) For each $p \in \mathcal{C}_{\mathcal{B}}$, $g_{\mathcal{B}}(f(p)) = p$.

(iii) For each $\sigma \in \mathcal{L}_{\mathcal{B}}$, $f(g_{\mathcal{B}}(\sigma))$ is a subsequence of σ.

Proof. Part (i): We first show that f maps $\mathcal{C}_{\mathcal{B}}$ into $\mathcal{L}_{\mathcal{B}}$. Let $p \in \mathcal{C}_{\mathcal{B}}$ and let $f(p) = \mathcal{C}_{\mathcal{B}}$. Since the sets $V_0, \ldots, V_{n-1}$ belong to $\mathcal{B}$ and form a partition of Ω, each atom B of $\mathcal{B}$ is contained in exactly one of the sets V_i. By the definition of $f(p)$ we have $\mu_j(B) > 0$ if and only if $j = i$. Thus $f(p) \in \mathcal{L}_{\mathcal{B}}$.

To show that f is injective, suppose $p, q \in \mathcal{C}_{\mathcal{B}}$ and $f(p) = f(q) = (\mu_0, \ldots, \mu_n)$. Then the partitions $V_0, \ldots, V_{n-1}$ must be the same for p and q, because V_i is the union of all atoms B such that $\mu_i(B) > 0$. Let $C \in \mathcal{B}$, take the least i such that $V_i \cap C \neq \emptyset$, and let K be the set of atoms of $\mathcal{B}$ contained in $V_i \cap C$. Then for any set $A \in \mathcal{A}$ with $A \subseteq C$, we have

$$p(A \mid C) = \frac{p(A \mid V_i)}{p(C \mid V_i)} = \frac{p(A \mid V_i)}{\sum_{B \in K} p(B \mid V_i)} = \frac{\mu_i(A)}{\sum_{B \in K} \mu_i(B)}.$$

The same formula also holds for $q(A \mid C)$, and therefore $p = q$.

Part (ii): Let $f(p) = (\mu_0, \ldots, \mu_{n-1})$ and note that, for each atom A of $\mathcal{B}$, there is an i with $\mu_i(A) > 0$. So, for any event B in $\mathcal{B}$, there is a minimum i with $\mu_i(B) > 0$. For this i, we have $\mu_i(B) = p(B|W_i)$. Then, for any event E,

$$g_{\mathcal{B}}(f(p))(E \cap B|W_i) = g_{\mathcal{B}}(f(p))(E \cap B|B)\, g_{\mathcal{B}}(f(p))(B|W_i)$$

or

$$g_{\mathcal{B}}(f(p))(E|B) = \frac{\mu_i(E \cap B)}{\mu_i(B)}.$$

But note that

$$p(E|B) = \frac{\mu_i(E \cap B)}{\mu_i(B)}.$$

Part (iii): Let $\sigma = (\nu_0, \ldots, \nu_{n-1})$. From the definitions of f and $g_{\mathcal{B}}$, $f(g_{\mathcal{B}}(\sigma))$ is the subsequence of σ consisting of those ν_i such that for some atom B of $\mathcal{B}$, i is the least integer with $\nu_i(B) > 0$. ■

3.2 From Assumption to Strong Belief

Definition 11 *Say $\mathcal{B}$ is **open** if each $B \in \mathcal{B}$ is open.*

Proposition 6 *Fix some $\mathcal{B}$ that is open. If $\sigma \in \mathcal{L}_{\mathcal{B}}$ and E is assumed under σ then E is strongly believed under $g_{\mathcal{B}}(\sigma)$.*

Proof. Suppose E is assumed under $\sigma = (\nu_0, \ldots, \nu_{n-1})$ at level j. Let $B \in \mathcal{B}$ with $E \cap B \neq \emptyset$. By condition a of assumption, for each $i \leq j$, $\nu_i(B) = \nu_i(B \cap E)$. So, using conditions b-c of assumption, there is some $i \leq j$ with $\nu_i(B) > 0$. Let k be the minimum i with $\nu_i(B) > 0$. Then $g_{\mathcal{B}}(\sigma)(\cdot|B) = \nu_k(\cdot|B)$. We also have

$$\nu_k(E|B) = \frac{\nu_k(E \cap B)}{\nu_k(B)} = 1,$$

so that $g_{\mathcal{B}}(\sigma)(E|B) = \nu_k(E|B) = 1$ as required. ■

The purpose of this next example is to show that Proposition 6 cannot be improved to say: An E is assumed under σ if and only if it is strongly believed under $g_{\mathcal{B}}(\sigma)$.

Example 6 *Take $\Omega = (0,2)$ and $\sigma = (\nu_0, \nu_1, \nu_2)$ where ν_0 is uniform on $(0,1)$, $\nu_1(1) = 1$, and ν_2 is uniform on $(1,2)$. Let $E = (0,1) \cup (1,2)$. Then E is not assumed under σ. But, for any open $\mathcal{B}$, $g_{\mathcal{B}}(\sigma)$ strongly believes E. To see this, fix an open $\mathcal{B}$ and some $B \in \mathcal{B}$ with $B \cap [(0,1) \cup (1,2)] \neq \emptyset$. Write $U_1 = B \cap (0,1)$ and $U_2 = B \cap (1,2)$. Suppose $U_1 \neq \emptyset$. Then $\nu_0(B) \geq \nu_0(U_1) > 0$, so*

$$g_{\mathcal{B}}(\sigma)(E|B) = \frac{\nu_0([(0,1) \cup (1,2)] \cap B)}{\nu_0(B)} = 1,$$

as required. If $U_1 = \emptyset$ then $U_2 \neq \emptyset$ and again $\nu_2(B) \geq \nu_2(U_2) > 0$, so

$$g_{\mathcal{B}}(\sigma)(E|B) = \frac{\nu_2([(0,1) \cup (1,2)] \cap B)}{\nu_2(B)} = 1,$$

as required.

We now mention two limited converses to Proposition 6. Here is one, that deals with the case when $\mathcal{B}$ is the collection of all non-empty open sets.

Lemma 9 *Let $\mathcal{B}$ be the set of all non-empty open sets and let $\sigma \in \mathcal{L}_{\mathcal{B}}$. Suppose E is strongly believed under $g_{\mathcal{B}}(\sigma)$ and E satisfies conditions a-b of assumption for some j. Then σ assumes E.*

Proof. Fix some $\sigma \in \mathcal{L}_{\mathcal{B}}$ satisfying the premise of the lemma. Let U be an open set with $E \cap U \neq \emptyset$. Then $U \in \mathcal{B}$, so that there exists some i with $\mu_i(U \cap E) = \mu_i(E) > 0$. This establishes condition c of assumption. ∎

Taken together, Proposition 6 and Lemma 9 establish the following.

Corollary 2 *Let $\mathcal{B}$ be the collection of all non-empty open sets. Suppose $\sigma \in \mathcal{L}_{\mathcal{B}}$ and E is an event satisfying conditions a-b of assumption. Then E is assumed under σ if and only if E is strongly believed under $g_{\mathcal{B}}(\sigma)$.*

Lemma 10 *Let $\mathcal{B}$ be open. Then every LPS $\sigma \in \mathcal{L}$ with full support belongs to $\mathcal{L}_{\mathcal{B}}$.*

Proof. Let $x \in B \in \mathcal{B}$. For some i we have $x \in \operatorname{Supp} \mu_i$. Since B is open, $\mu_i(B) > 0$. Therefore σ covers $\mathcal{B}$. ∎

Example 7 *The purpose of this example is to show that, if $\mathcal{B}$ is a finite algebra but not open, an event may be assumed under $f(p)$ even though it is not strongly believed under p.*

Take $\Omega = [0,2]$ and let $\mathcal{B}$ be generated by the partition $\{\{0\}, (0,1), [1,2), \{2\}\}$. Also, choose a CPS p so that $p(\cdot|\Omega) = p(\cdot|[1,2))$ is uniform on $[1,2)$, $p(\cdot|(0,1))$ is uniform on $(0,1)$, and $p(0|\{0,2\}) = p(2|\{0,2\}) = \frac{1}{2}$. Then $f(p) = (\mu_0, \mu_1)$ is such that μ_0 is uniform on $[1,2)$ and μ_1 is uniform on $(0,1)$. Note, $\operatorname{Supp} \mu_0 = [1,2]$ and $\operatorname{Supp} \mu_1 = [0,1]$, so that $[1,2]$ is assumed under $f(p)$. But $[1,2] \cap \{0,2\} \neq \emptyset$ and $p([1,2] || \{0,2\}) = \frac{1}{2}$.

For the second, we will look at a set of conditions that are both a finite algebra and open. This implies that each conditioning event is clopen. We begin by investigating properties of this case. Recall from Proposition 2 that full support for a CPS p implies full support for $f(p)$. The next result shows that the converse also holds when $\mathcal{B}$ is an open finite algebra.

Proposition 7 *Suppose $\mathcal{B}$ is an open finite algebra. If $p \in \mathcal{C}_{\mathcal{B}}$ and $f(p)$ has full support then p has full support.*

Proof. Let $f(p) = (\mu_0, \ldots, \mu_{n-1})$, let B be an atom of $\mathcal{B}$, and let U be a nonempty open subset of B. We must show that $p(U \mid B) > 0$.

By Lemma 3, there is a unique $i < n$ such that $B \subseteq V_i$. Since V_i is clopen and $\mu_j(V_i) = 0$ when $j \neq i$, $\operatorname{Supp} \mu_j \cap V_i = \emptyset$ when $j \neq i$. Therefore $V_i \subseteq \operatorname{Supp} \mu_i$. By definition, $\mu_i = p(\cdot \mid V_i)$, so $V_i \subseteq \operatorname{Supp} p(\cdot \mid V_i)$. It follows that $p(U \mid V_i) > 0$. Since $U \subseteq B \subseteq V_i$, we have $p(U \mid V_i) = p(U \mid B) p(B \mid V_i)$, and hence $p(U \mid B) > 0$ as required. ∎

Corollary 3 *Suppose $\mathcal{B}$ is an open finite algebra. If $p \in \mathcal{C}_{\mathcal{B}}$ and E is assumed under $f(p)$, then E is strongly believed under p.*

Proof. By Proposition 5 we have $p = g_{\mathcal{B}}(f(p))$, and by Proposition 6, E is strongly believed under $g_{\mathcal{B}}(f(p))$. ■

Corollary 4 *Fix an open finite algebra $\mathcal{B}$ and some $p \in \mathcal{C}_{\mathcal{B}}$ that has full support. Then E is assumed under $f(p)$ if and only if E is strongly believed under p.*

Proof. By Proposition 3 and Corollary 3. ■

Lemma 11 *Let $\mathcal{B}$ be an open finite algebra. Then $g_{\mathcal{B}}$ maps the set of full-support LPS's surjectively onto $\mathcal{C}_{\mathcal{B}}$.*

Proof. Take $p \in \mathcal{C}_{\mathcal{B}}$ and let $f(p) = (\mu_0, \ldots, \mu_{n-1})$. Let $S = \bigcup_{i<n} \operatorname{Supp} \mu_i$. If $S = \Omega$ we are done. Suppose $S \neq \Omega$. Then $\Omega \setminus S$ is a nonempty open set. Since Ω is Polish, $\Omega \setminus S$ has a countable dense subset U. We may choose a probability measure μ_n such that $\mu_n(U) = 1$ and $\mu_n(\{a\}) > 0$ for each $a \in U$. Then $\tau = (\mu_0, \ldots, \mu_n)$ is an LPS with full support, and $g_{\mathcal{B}}(\tau) = g_{\mathcal{B}}(f(p)) = p$. ■

Proposition 8 *Suppose $\mathcal{B}$ is an open finite algebra. Fix $p \in \mathcal{C}_{\mathcal{B}}$. Then a set E is strongly believed under p if and only if E is assumed under some full-support LPS τ such that $g_{\mathcal{B}}(\tau) = p$.*

Proof. One direction follows from Proposition 6 and Lemma 10.

To prove the other direction, suppose that E is strongly believed under p. Let

$$f(p) = (\mu_0, \ldots, \mu_{n-1}).$$

Then for some $j < n$, $E \subseteq \bigcup_{i=0}^{j} V_i$ and $\mu_i(E \cap V_i) = 1$ for all $i \leq j$. We will construct an LPS τ which is a sequence of the form

$$\tau = (\mu_0, \nu_0, \ldots, \mu_j, \nu_j, \mu_{j+1}, \ldots, \mu_{n-1}, \kappa),$$

where some of the ν_i and κ may be missing. The idea is that ν_i will capture the part of $E \cap V_i$ which is outside the support of μ_i, and κ will capture the part of Ω outside the supports of the rest of τ. For $i \leq j$, if $E \cap V_i \subseteq \operatorname{Supp} \mu_i$, then ν_i is missing. Otherwise, take a countable dense subset U_i of $E \cap V_i \setminus \operatorname{Supp} \mu_i$, and let ν_i be a measure such that $\nu_i(U_i) = 1$ and ν_i gives positive measure to each point of U_i. Now let

$$Y = \bigcup_{i=0}^{n-1} \operatorname{Supp} \mu_i \cup \bigcup_{i=0}^{j} \operatorname{Supp} \nu_i.$$

If $Y = \Omega$, κ is missing. Otherwise take a countable dense subset Z of $\Omega \setminus Y$, and let κ be a measure such that $\kappa(Z) = 1$ and κ gives positive measure to each point of Z. Since each U_i is outside the support of μ_i, τ is a sequence of mutually singular measures, and hence is an LPS. We have $Z \subseteq \operatorname{Supp} \kappa$, so τ has full support. Also, $E \subseteq \bigcup_{i=0}^{j} (\operatorname{Supp} \mu_i \cup \operatorname{Supp} \nu_i)$. Therefore E is assumed under τ. Finally, $g_{\mathcal{B}}(\tau) = p$, because for each i and each atom $B \subseteq V_i$, μ_i is the first measure in the sequence τ which gives positive measure to B. ■

References

[1] Asheim, G., and Y. Søvik, "Preference-Based Belief Operators," *Mathematical Social Sciences*, 50, 2005, 61-82.

[2] Battigalli, P., "Interactive Epistemology in the Extensive and Normal Form," unpublished, 2004.

[3] Battigalli, P., and M. Siniscalchi, "Strong Belief and Forward-Induction Reasoning," *Journal of Economic Theory*, 106, 2002, 356-391.

[4] Blume, L., A. Brandenburger, and E. Dekel, "An Overview of Lexicographic Choice Under Uncertainty," *Annals of Operations Research*, 19, 1989, 231-246.

[5] Brandenburger, A., A., Friedenberg, and H.J. Keisler, "Admissibility in Games," unpublished, 2004. Available at www.stern.nyu.edu/∼abranden.

[6] Halpern, J., "Lexicographic Probability, Conditional Probability, and Nonstandard Probability," unpublished, 2003. Available at www.cs.cornell.edu/home/halpern.

[7] Hammond, P., "Elementary Non-Archimedean Representations of Probability for Decision Theory and Games," in P. Humphreys (ed.), *Patrick Suppes: Scientific Philosopher, Vol. I: Probability and Probabilistic Causality*, Kluwer, 1994, 25-59.