The Relationship Between Rationality on the Matrix and the Tree*

Adam Brandenburger† Amanda Friedenberg‡

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The relationship between the matrix and the tree has been the subject of intensive investigation ever since the beginning of game theory. The issue goes back even to Borel and von Neumann.

Later, Thompson (1997), followed by Elmes and Reny (1994), uncovered the structural relationship between the matrix and the tree. They showed that, up to the duplication of pure strategies, two games have the same strategic form if and only if they differ by a certain sequence of elementary transformations.

There is also the question of the relationship between Nash equilibrium defined on the matrix (perfect, proper, stable equilibria, etc.) and equilibrium defined on the tree (extensive-form perfect, sequential equilibria, etc.). For example, Kohlberg and Mertens (1986) and Van Damme (1984) showed that a strategic-form proper equilibrium induces a sequential-equilibrium outcome in any tree with that strategic form.

But a very basic question has remained: What is the relationship between dominance in the matrix and dominance in the tree? And, following from this, what is the relationship between iterated dominance in the matrix and iterated dominance in the tree? This note addresses this question. See Shimoji (2004) for a related analysis. analyses.

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†adam.brandenburger@stern.nyu.edu, http://pages.stern.nyu.edu/\protect\unhbox\voidb@x\penalty\@M\{}abranden/

‡amanda.friedenberg@asu.edu, http://www.public.asu.edu/\protect\unhbox\voidb@x\penalty\@M\{}afrieden/

1 Set-Up

For a finite set X, let $\mathcal{M}(X)$ be the set of probability measures on X and $\mathcal{M}^+(X)$ the set of full-support measures on X. Given a set $Y \subseteq X$, we will often identify $\mathcal{M}(Y)$ (resp. $\mathcal{M}^+(Y)$) with the set of probability measures on X with support contained in (resp. equal to) Y.

We begin with a finite **extensive-form game** Γ between Ann and Bob.[1] Let S^a (resp. S^b) be the set of strategies for Ann (resp. Bob). Note that S^a and S^b are finite. Let H^a (resp. H^b) be the family of information sets at which Ann (resp. Bob) moves, and let $H = H^a \cup H^b$. Write $S^a(h)$ (resp. $S^b(h)$) for the set of Ann's strategies that allow information set h. Let Z be the set of terminal nodes, and $\zeta : S^a \times S^b \to Z$ map each strategy profile to the terminal node it reaches. Extensive-form payoff functions are maps $\Pi^a :\to \mathbb{R}$ and $\Pi^b :\to \mathbb{R}$.

We restrict attention to extensive-form games with perfect recall (Kuhn, 1950, 1953). These games satisfy an important property. In perfect-recall games, we have: For all information sets h and i, either $S^a(h) \subseteq S^a(i)$, $S^a(i) \subseteq S^a(h)$, or $S^a(h) \cap S^a(i) = \emptyset$.

An extensive-form game Γ induces a **strategic-form game** $G = \langle S^a, S^b, \pi^a, \pi^b \rangle$, where $\pi^a = \Pi^a \circ \zeta$ and $\pi^b = \Pi^b \circ \zeta$. We extend π^a to $\mathcal{M}(S^a) \times \mathcal{M}(S^b)$ in the usual way, i.e. $\pi^a(\sigma^a, \sigma^b) = \sum_{s^a \in S^a} \sum_{s^b \in S^b} \pi^a(s^a, s^b)\sigma^b(s^b)\sigma^a(s^a)$.

The following definitions all have counterparts with a and b reversed.

Definition 1.1 *Fix $Y^a \times Y^b \subseteq S^a \times S^b$. A strategy $s^a \in Y^a$ is **(strongly) dominated with respect to** $Y^a \times Y^b$ if there exists $\sigma^a \in \mathcal{M}(Y^a)$ such that $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for every $s^b \in Y^b$. Otherwise, say s^a is **undominated with respect to** $Y^a \times Y^b$. If s^a is undominated with respect to $S^a \times S^b$, simply say that s^a is **undominated**.*

Definition 1.2 *Fix $Y^a \times Y^b \subseteq S^a \times S^b$. A strategy $s^a \in Y^a$ is **weakly dominated with respect to** $Y^a \times Y^b$ if there exists $\sigma^a \in \mathcal{M}(Y^a)$ such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y^b$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y^b$. Otherwise, say s^a is **admissible with respect to** $Y^a \times Y^b$. If s^a is admissible with respect to $S^a \times S^b$, simply say that s^a is **admissible**.*

Note, if s^a is not contained in Y^a, then s^a is undominated (resp. admissible) given $Y^a \times Y^b$.

We have the usual equivalences:

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[1] The restriction to two-player games is for notational simplicity only.

Lemma 1.1 *A strategy* $s^a \in Y^a$ *is undominated with respect to* $Y^a \times Y^b$ *if and only if there exists* $\sigma^b \in \mathcal{M}(Y^b)$ *such that* $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ *for every* $r^a \in Y^a$.

Lemma 1.2 *A strategy* $s^a \in Y^a$ *is admissible with respect to* $Y^a \times Y^b$ *if and only if there exists* $\sigma^b \in \mathcal{M}^+(Y^b)$ *such that* $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ *for every* $r^a \in Y^a$.

We now define six procedures. The first two procedures are on the matrix. Let $G_0^a \times G_0^b = G_0^{+,a} \times G_0^{+,b} = S^a \times S^b$. Define $G_m^a \times G_m^b$ by induction, where G_{m+1}^a is the set of strategies $s^a \in G_m^a$ are undominated with respect to $G_m^a \times G_m^b$. And likewise with a and b reversed. Define $G_m^{+,a} \times G_m^{+,b}$ by induction, where $G_{m+1}^{+,a}$ is the set of strategies $s^a \in G_m^{+,a}$ that are admissible with respect to $G_m^{+,a} \times G_m^{+,b}$. And likewise with a and b reversed.

Definition 1.3 *Say* s^a *is* ***m-rationalizable*** *if* $s^a \in G_m^a$. *Say* s^a *is* **m*-admissible*** *if* $s^a \in G_m^{+,a}$.

The remaining procedures are defined on the tree. Let

$$\Gamma_0^a \times \Gamma_0^b = \Gamma_0^{+,a} \times \Gamma_0^{+,b} = \hat{\Gamma}_0^a \times \hat{\Gamma}_0^b = \hat{\Gamma}_0^{+,a} \times \hat{\Gamma}_0^{+,b} = S^a \times S^b.$$

Define $\Gamma_m^a \times \Gamma_m^b$ (resp. $\hat{\Gamma}_m^a \times \hat{\Gamma}_m^b$) by induction, where Γ_{m+1}^a (resp. $\hat{\Gamma}_{m+1}^a$) is the set of all strategies s^a in Γ_m^a (resp. $\hat{\Gamma}_m^a$) so that, for each $h \in H^a$ (resp. $h \in H$), s^a is undominated with respect to $[\Gamma_m^a \cap S^a(h)] \times [\Gamma_m^b \cap S^b(h)]$ (resp. $[\hat{\Gamma}_m^a \cap S^a(h)] \times [\hat{\Gamma}_m^b \cap S^b(h)]$). And, likewise with a and b reversed. Define $\Gamma_m^{+,a} \times \Gamma_m^{+,b}$ (resp. $\hat{\Gamma}_m^{+,a} \times \hat{\Gamma}_m^{+,b}$) by induction, where $\Gamma_{m+1}^{+,a}$ (resp. $\hat{\Gamma}_{m+1}^{+,a}$) is the set of all strategies s^a in $\Gamma_m^{+,a}$ (resp. $\hat{\Gamma}_m^{+,a}$) so that, for each $h \in H^a$ (resp. $h \in H$), s^a is admissible with respect to $[\Gamma_m^{+,a} \cap S^a(h)] \times [\Gamma_m^{+,b} \cap S^b(h)]$ (resp. $[\hat{\Gamma}_m^{+,a} \cap S^a(h)] \times [\hat{\Gamma}_m^{+,b} \cap S^b(h)]$). And, likewise with a and b reversed.

Definition 1.4 *Say* s^a *is* m***-extensive-form rationalizable*** *if* $s^a \in \Gamma_m^a$. *Say* s^a *is* m***-extensive-form admissible*** *if* $s^a \in \Gamma_m^{+,a}$.

The concept of extensive-form rationalizability is due to Pearce (1984). (See, also, Battigalli, 1997.) Papers often amend this procedure and consider the procedure of $\hat{\Gamma}_m^a \times \hat{\Gamma}_m^b$, for technical simplicity. See, e.g., Battigalli and Siniscalchi (2002).

2 Summary of Results

It is well known that:

$$G_m^{+,a} \times G_m^{+,b} \subseteq G_m^a \times G_m^b,$$

for each m. In Section 3 we show:

$$G_m^{+,a} \times G_m^{+,b} = \Gamma_m^{+,a} \times \Gamma_m^{+,b} = \hat{\Gamma}_m^{+,a} \times \hat{\Gamma}_m^{+,b},$$

for each m. In Section 4, we show that under a condition on the tree we call No Relevant Convexities:

$$\hat{\Gamma}_m^{a} \times \hat{\Gamma}_m^{b} = \hat{\Gamma}_m^{+,a} \times \hat{\Gamma}_m^{+,b}.$$

Finally, in Section 5, we show that

$$\hat{\Gamma}_m^{a} \times \hat{\Gamma}_m^{b} = \Gamma_m^{a} \times \Gamma_m^{b},$$

for each m. It follows that, in games satisfying No Relevant Convexities:

$$\Gamma_m^{a} \times \Gamma_m^{b} = \Gamma_m^{+,a} \times \Gamma_m^{+,b} = G_m^{+,a} \times G_m^{+,b}.$$

3 Admissibility in the Matrix and the Tree

Proposition 3.1 *Fix an extensive-form game* Γ *with associated strategic form* G. *Then* $G_m^{+,a} \times G_m^{+,b} = \Gamma_m^{+,a} \times \Gamma_m^{+,b} = \hat{\Gamma}_m^{+,a} \times \hat{\Gamma}_m^{+,b}$ *for all* m.

The proposition will follow immediately from:

Lemma 3.1 *The following are equivalent:*

(i) s^a *is admissible with respect to* $Y^a \times Y^b$.

(ii) For each $h \in H^a$, s^a *is admissible with respect to* $(Y^a \cap S^a(h)) \times (Y^b \cap S^b(h))$.

(iii) For each $h \in H$, s^a *is admissible with respect to* $(Y^a \cap S^a(h)) \times (Y^b \cap S^b(h))$.

Proof. We first show that part (i) implies part (iii). Certainly, part (iii) implies part (ii). Finally, we show part (ii) implies part (i).

Part (i) implies Part (iii): Fix s^a admissible with respect to $Y^a \times Y^b$. It suffices to consider the case where $s^a \in Y^a$. Then, there is a measure $\sigma^b \in \mathcal{M}^+(Y^b)$ with $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for all $r^a \in Y^a$. Fix an information set $h \in H$ with $s^a \in S^a(h)$. We will show that if $Y^b \cap S^b(h) \neq \emptyset$, then $\pi^a(s^a, \sigma^b(\cdot|S^b(h))) \geq \pi^a(r^a, \sigma^b(\cdot|S^b(h)))$ for all $r^a \in Y^a \cap S^a(h)$.

(Note, in this case, $\sigma^b(\cdot|S^b(h))$ is well defined, since $Y^b \cap S^b(h) \neq \emptyset$ implies $\sigma^b(Y^b \cap S^b(h)) > 0$.)

Suppose not. Then there is an $r^a \in S^a(h)$ with $\pi^a(r^a, \sigma^b(\cdot|S^b(h))) > \pi^a(s^a, \sigma^b(\cdot|S^b(h)))$. Since $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$, we must have $\sigma^b(S^b(h)) < 1$. Let q^a be the strategy that agrees with r^a at h onwards but otherwise agrees with s^a. We have

$$\begin{aligned}
\pi^a(q^a, \sigma^b) &= \sigma^b(S^b(h))\pi^a(r^a, \sigma^b(\cdot|S^b(h))) + (1 - \sigma^b(S^b(h)))\pi^a(s^a, \sigma^b(\cdot|S^b(h))) \\
&> \sigma^b(S^b(h))\pi^a(s^a, \sigma^b(\cdot|S^b(h))) + (1 - \sigma^b(S^b(h)))\pi^a(s^a, \sigma^b(\cdot|S^b(h))) \\
&= \pi^a(s^a, \sigma^b),
\end{aligned}$$

a contradiction.

Part (ii) implies Part (i): Suppose that, for each information set $h \in H^a$, s^a is (extensive-form) admissible with respect to $[Y^a \cap S^a(h)] \times [Y^b \cap S^b(h)] \neq \emptyset$. If s^a is not in Y^a, then certainly s^a is admissible with respect to $Y^a \times Y^b$. So, we suppose $s^a \in Y^a$.

Note, we can find information sets $1, .., K \in H^a$ so that: (i) $S^a = S^a(k)$ for each $k = 1, .., K$; and (ii) the sets $S^b(1), \ldots, S^b(K)$ form a partition of S^b: If Ann moves at the initial node, simply take $K = 1$. If Bob moves at the initial node, each of Bob's moves at the initial node, viz. $1, \ldots, C$ leads to an information set. (Of course, some of these moves may lead to the same information set.) If the choice c leads to an information set of Bob's, then we go one level further. Eventually, we reach a collection of information sets for Ann satisfying the desired properties.

Fix $J \leq K$ so that $Y^b \cap S^b(k) \neq \emptyset$ for all $1 \leq k \leq J$ and $Y^b \cap S^b(k) = \emptyset$ for all $J < k \leq K$. For each $k \leq J$, there is a measure $\sigma^b_k \in \mathcal{M}^+(Y^b \cap S^b(k))$ with $\pi^a(s^a, \sigma^b_k) \geq \pi^a(r^a, \sigma^b_k)$ for all $r^a \in Y^a$. Build σ^b so that, for each $s^b \in Y^b$, $\sigma^b(s^b) = \frac{1}{J}\sigma^b_k(s^b)$, where $s^b \in S^b(k)$. It is immediate that this defines a probability measure in $\mathcal{M}^+(Y^b)$, and, moreover,

$$\pi^a(q^a, \sigma^b) = \frac{1}{J}\sum_{k=1}^{J} \pi^a(q^a, \sigma^b_k),$$

for each $q^a \in Y^a$. Therefore $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for all $r^a \in Y^a$. This establishes that s^a is admissible with respect to $Y^a \times Y^b$. ■

Proof of Proposition 3.1. The result is immediate for $m = 0$. Assuming the result for $m \geq 1$ and applying Lemma 3.1 gives the result for $m + 1$. ■

4 No Relevant Convexities

Definition 4.1 *Say* r^a ***supports*** $\mathbf{s^a}$ ***with respect to*** $\mathbf{Y^b} \subseteq \mathbf{S^b}$ *if there exists* $\sigma^a \in \mathcal{M}(S^a)$ *such that*

(i) $r^a \in \operatorname{Supp} \sigma^a$*, and*

(ii) $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ *for all* $s^b \in Y^b$*.*

Definition 4.2 *An extensive-form game* Γ *satisfies* ***No Relevant Convexities (NRC)*** *if whenever* r^a *supports* s^a *with respect to some* $Y^b \subseteq S^b$*, then* $\zeta(s^a, s^b) = \zeta(r^a, s^b)$ *for each* $s^b \in Y^b$*.*

The term NRC is a strengthening of No Relevant Ties, due to Battigalli (1997). (An NRC game satisfies No Relevant Ties.)

Proposition 4.1 *Fix an extensive-form game* Γ *satisfying NRC. Then* $\hat{\Gamma}^a_m \times \hat{\Gamma}^b_m = \hat{\Gamma}^{+,a}_m \times \hat{\Gamma}^{+,b}_m$ *for all* m*.*

The proof will make use of the following implication of NRC.

Lemma 4.1 *Fix an extensive-form game* Γ *satisfying NRC and some* $Y^a \times Y^b \subseteq S^a \times S^b$*. Then the following are equivalent:*

(a) The strategy s^a *is undominated with respect to* $Y^a \times Y^b$*.*

(b) There exists $\sigma^b \in \mathcal{M}(Y^b)$ *with*

  (a) $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ *for all* $r^a \in Y^a$*;*

  (b) if $r^a \in Y^a$ *satisfies* $\pi^a(r^a, \sigma^b) = \pi^a(s^a, \sigma^b)$*, then* $\zeta(r^a, s^b) = \zeta(r^a, s^b)$ *for every* $s^b \in \operatorname{Supp} \sigma^b$*.*

Proof. Certainly (b) implies (a) (for any game). We will show (a) implies (b).

Suppose s^a is undominated with respect to $Y^a \times Y^b$. Then, s^a must be admissible with respect to $Y^a \times Z^b$ for some $Z^b \subseteq Y^b$. Lemma D.4 in Brandenburger, Friedenberg and Keisler (2008), we can find a $\sigma^b \in \mathcal{M}^+(Z^b)$ so that: r^a supports s^a with respect to Z^b if and only if $\pi^a(r^a, \sigma^b) \geq \pi^a(q^a, \sigma^b)$ for all $q^a \in Y^a$.

Certainly $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for all $r^a \in Y^a$. Suppose there is some $r^a \in Y^a$ with $\pi^a(s^a, \sigma^b) = \pi^a(r^a, \sigma^b)$. Then, r^a supports s^a with respect to $Z^b = \operatorname{Supp} \sigma^b$. So, by NRC, $\zeta(s^a, s^b) = \zeta(r^a, s^b)$ for each $s^b \in \operatorname{Supp} \sigma^b$. ■

Proof. The proof is by induction on m. The result is immediate for $m = 0$. Assume the result holds for $m \geq 1$. By the induction hypothesis, $\hat{\Gamma}^{+,a}_{m+1} \times \hat{\Gamma}^{+,b}_{m+1} \subseteq \hat{\Gamma}^{a}_{m+1} \times \hat{\Gamma}^{b}_{m+1}$. We will show that $\hat{\Gamma}^{a}_{m+1} \times \hat{\Gamma}^{b}_{m+1} \subseteq \hat{\Gamma}^{+,a}_{m+1} \times \hat{\Gamma}^{+,b}_{m+1}$.

Fix $s^a \in \hat{\Gamma}^{a}_{m+1}$. Let $\overline{H}$ be the family of information sets $h \in H$ allowed by $\{s^a\} \times \hat{\Gamma}^{b}_{m}$. It suffices to show that, for each $h \in \overline{H}$, s^a is admissible with respect to $\hat{\Gamma}^{a}_{m} \times \hat{\Gamma}^{b}_{m}$. If so, the result follows from the induction hypothesis.

Suppose not. Then, there is an information set $h \in \overline{H}$ such that s^a is inadmissible with respect to $[\hat{\Gamma}^{a}_{m} \cap S^a(h)] \times [\hat{\Gamma}^{b}_{m} \cap S^b(h)]$. In particular, pick h so that, for each $i \in \overline{H}$ the following holds: For each $i \in \overline{H}$ with $S(i) \subsetneq S(h)$, s^a is admissible with respect to $[\hat{\Gamma}^{a}_{m} \cap S^a(i)] \times [\hat{\Gamma}^{b}_{m} \cap S^b(i)]$.)

Since $s^a \in \hat{\Gamma}^{a}_{m+1}$, s^a is undominated given $[\hat{\Gamma}^{a}_{m} \cap S^a(h)] \times [\hat{\Gamma}^{b}_{m} \cap S^b(h)]$. By Lemma **??**, there exists $\sigma^b \in \mathcal{M}(\hat{\Gamma}^{b}_{m} \cap S^b(h))$ with

(i) $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for all $r^a \in [\hat{\Gamma}^{a}_{m} \cap S^a(h)]$;

(ii) if $r^a \in [\hat{\Gamma}^{a}_{m} \cap S^a(h)]$ satisfies $\pi^a(r^a, \sigma^b) = \pi^a(s^a, \sigma^b)$, then $\zeta(r^a, s^b) = \zeta(r^a, s^b)$ for every $s^b \in \operatorname{Supp} \sigma^b$.

It suffices to show the following claim:

Claim 4.1 *If* s^a *is inadmissible given* $[\hat{\Gamma}^{a}_{m} \cap S^a(h)] \times [\hat{\Gamma}^{b}_{m} \cap S^b(h) \backslash \operatorname{Supp} \sigma^b]$, *then each* $r^a \in [\hat{\Gamma}^{a}_{m} \cap S^a(h)]$ *with* $\pi^a(r^a, \sigma^b) = \pi^a(s^a, \sigma^b)$ *is also inadmissible given* $[\hat{\Gamma}^{a}_{m} \cap S^a(h)] \times [\hat{\Gamma}^{b}_{m} \cap S^b(h) \backslash \operatorname{Supp} \sigma^b]$.

If so, then we can find a a measure $\rho^b \in \mathcal{M}^+((\hat{\Gamma}^{b}_{m} \cap S^b(h)) \backslash \operatorname{Supp} \sigma^b)$ so that $\pi^a(s^a, \rho^b) \geq \pi^a(r^a, \rho^b)$ for each $r^a \in [\hat{\Gamma}^{a}_{m} \cap S^a(h)]$ with $\pi^a(r^a, \sigma^b) = \pi^a(s^a, \sigma^b)$. With this, construct a measure $\varpi^b in \mathcal{M}^+(\hat{\Gamma}^{b}_{m} \cap S^b(h))$ by setting $\varpi^b(s^b) = (1-\epsilon)\sigma^b(s^b) + \epsilon\rho^b(s^b)$, where $\epsilon \in (0,1)$. Then note that, using the above claim, we can choose $\epsilon > 0$ sufficiently small so that $\pi^a(s^a, \varpi^b) \geq \pi^a(r^a, \varpi^b)$ for each $r^a \in [\hat{\Gamma}^{a}_{m} \cap S^a(h)]$. Given this, s^a is admissible given $[\hat{\Gamma}^{a}_{m} \cap S^a(h)] \times [\hat{\Gamma}^{b}_{m} \cap S^b(h)]$.

We now turn to show the claim. We take two cases:

Case 4.1 *Ann moves at h.*

, i.e., there exists some σ^b

Since s^a is inadmissible given $[\hat{\Gamma}^{a}_{m} \cap S^a(h)] \times [\hat{\Gamma}^{b}_{m} \cap S^b(h)]$, there exists some $\sigma^a \in \hat{\Gamma}^{a}_{m} \cap S^a(h)$ so that

(i) $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for each $s^b \in \hat{\Gamma}^b_m \cap S^b(h)$,

(ii) $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in \hat{\Gamma}^b_m \cap S^b(h)$,

Since s^a is undominated given $[\hat{\Gamma}^a_m \cap S^a(h)] \times [\hat{\Gamma}^b_m \cap S^b(h)]$, it follows that there exists

$$Z^b = \{s^b \in \hat{\Gamma}^b_m \cap S^b(h)\}$$

$Z^b \neq \emptyset$ so that, for each

■

Proof of Proposition 4.1. The proof is by induction on m. The result is immediate for $m = 0$. Assume the result holds for $m \geq 1$. By the induction hypothesis, $\hat{\Gamma}^{+,a}_{m+1} \times \hat{\Gamma}^{+,b}_{m+1} \subseteq \hat{\Gamma}^a_{m+1} \times \hat{\Gamma}^b_{m+1}$. We will show that $\hat{\Gamma}^a_{m+1} \times \hat{\Gamma}^b_{m+1} \subseteq \hat{\Gamma}^{+,a}_{m+1} \times \hat{\Gamma}^{+,b}_{m+1}$.

Fix $s^a \in \hat{\Gamma}^a_{m+1}$. Let $\overline{H}$ be the family of information sets $h \in H$ allowed by $\{s^a\} \times \hat{\Gamma}^b_m$. It suffices to show that, for each $h \in \overline{H}$, s^a is admissible with respect to $\hat{\Gamma}^a_m \times \hat{\Gamma}^b_m$. If so, the result follows from the induction hypothesis.

Suppose not. Then, there is an information set $h \in \overline{H}$ such that s^a is inadmissible with respect to $[\hat{\Gamma}^a_m \cap S^a(h)] \times [\hat{\Gamma}^b_m \cap S^b(h)]$. In particular, pick h so that, for each $i \in \overline{H}$ the following holds: If every node in i (strictly) follows some node in h, then s^a is admissible with respect to $[\hat{\Gamma}^a_m \cap S^a(i)] \times [\hat{\Gamma}^b_m \cap S^b(i)]$. (It follows that, for each $i \in \overline{H}$ with $S(i) \subsetneq S(h)$, s^a is admissible with respect to $[\hat{\Gamma}^a_m \cap S^a(i)] \times [\hat{\Gamma}^b_m \cap S^b(i)]$.)

Begin with the fact that $s^a \in \hat{\Gamma}^a_{m+1}$. By Lemma 4.1, there exists $\sigma^b_h \in \mathcal{M}(\hat{\Gamma}^b_m \cap S^b(h))$ such that:

(i) $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for all $r^a \in \hat{\Gamma}^a_m \cap S^a(h)$;

(ii) if $r^a \in \hat{\Gamma}^a_m \cap S^a(h)$ satisfies $\pi^a(r^a, \sigma^b) = \pi^a(s^a, \sigma^b)$, then $\zeta(r^a, s^b) = \zeta(r^a, s^b)$ for every $s^b \in \operatorname{Supp} \sigma^b$.

We divide the remainder of the proof into two cases. The first is where Ann moves at h and the second is where Bob moves at h.

Case 4.2 *Ann moves at h.*

Write $1, \ldots, C$ for the choices available to Ann at h. Each of these moves either leads to an information set (for either Ann or Bob) or a terminal history. Of course, two (or more) moves may lead to the same information set (in this case, of Bob). Write $1, \ldots, K$ for the information sets and note that $S^b(k) = S^b(h)$ for all $k = 1, \ldots, K$.

and note that $S^a(1), \ldots, S^a(K)$ form a partition of $S^a(h)$ and $S^b(k) = S^b(h)$ for all $k = 1, \ldots, K$. If $K = 1$ then we have found an information set K where each node in K strictly follows h and s^a is inadmissible with respect to $[\hat{\Gamma}^a_m \cap S^a(K)] \times [\hat{\Gamma}^b_m \cap S^b(K)]$, contradicting our choice of h. So, we must have $K > 1$.

Each move of Ann at h, leads to an information set of either Ann or Bob. (And, some moves may lead to the same information set.) Write $1, \ldots, K$ for these information sets and note that $S^a(1), \ldots, S^a(K)$ form a partition of $S^a(h)$ and $S^b(k) = S^b(h)$ for all $k = 1, \ldots, K$. If $K = 1$ then we have found an information set K where each node in K strictly follows h and s^a is inadmissible with respect to $[\hat{\Gamma}^a_m \cap S^a(K)] \times [\hat{\Gamma}^b_m \cap S^b(K)]$, contradicting our choice of h. So, we must have $K > 1$.

Without loss of generality, suppose that, at h, s^a makes the choice that leads to the information set $k = 1$.

Since s^a is, by assumption, inadmissible with respect to $[\hat{\Gamma}^a_m \cap S^a(h)] \times [\hat{\Gamma}^b_m \cap S^b(h)]$,

at h, we must have

Here, we can find information sets $1, \ldots, K$, so that $S^a(1), \ldots, S^a(K)$ form a partition of $S^a(h)$ and $S^b(k) = S^b(h)$ for all $k = 1, \ldots, K$. Note, if $K = 1$

Suppose there is a strategy r^a such that $\pi^a(s^a, \sigma^b_h) = \pi^a(r^a, \sigma^b_h)$. Then r^a must specify the same move as s^a at h. (This is property (ii) above.) By construction, s^a is admissible given $[\hat{\Gamma}^a_m \cap S^a(k)] \times [\hat{\Gamma}^b_m \cap S^b(k)]$. Since $S^b(k) = S^b(h)$, we have some measure $\rho^b_h \in \mathcal{M}^+(\hat{\Gamma}^b_m \cap S^b(h))$ with $\pi^a(s^a, \rho^b_h) \geq \pi^a(q^a, \rho^b_h)$ for all $q^a \in \hat{\Gamma}^a_m \cap S^a(k)$. Moreover, if $\pi^a(s^a, \sigma^b_h) = \pi^a(r^a, \sigma^b_h)$, then $\pi^a(s^a, \rho^b_h) \geq \pi^a(r^a, \rho^b_h)$. So, we can build a $\delta^b_h \in \mathcal{M}^+(\hat{\Gamma}^b_m \cap S^b(h))$ by setting $\delta^b_h(s^b) = (1 - \varepsilon)\sigma^b_h(s^b) + \varepsilon \rho^b_h(s^b)$, where $0 < \varepsilon < 1$. Then, there exists $\varepsilon > 0$ so that $\pi^a(s^a, \delta^b_h) \geq \pi^a(q^a, \delta^b_h)$ for all $q^a \in \hat{\Gamma}^a_m \cap S^a(h)$. (Here, we use the fact that if $\pi^a(s^a, \sigma^b_h) = \pi^a(r^a, \sigma^b_h)$, then $\pi^a(s^a, \rho^b_h) \geq \pi^a(r^a, \rho^b_h)$.) This contradicts the fact that s^a is inadmissible with respect to $[\hat{\Gamma}^a_m \cap S^a(h)] \times [\hat{\Gamma}^b_m \cap S^b(h)]$.

Choose a subset of $\overline{H}$, viz. $\hat{H}$, so that

(a) each $i \in \hat{H}$ follows h, i.e., $S(i) \subsetneq S(h)$;

(b) if j follows i and i follows h, i.e., $S(j) \subsetneq S(i) \subsetneq S(h)$, then

each $i \in \hat{H}$ follows h (i.e.,)

By construction, for each information set $i \in \overline{H}$ with $S(i) \subsetneq S(h)$, s^a is admissible with respect to $[\Gamma^a_m \cap S^a(i)] \times [\Gamma^b_m \cap S^b(i)]$.

Since, inadmissible with respect to $[\Gamma^a_m \cap S^a(h)] \times [\Gamma^b_m \cap S^b(h)]$, it follows that, for each $\rho^b_h \in \mathcal{M}^+(\Gamma^b_m \cap S^b(h))$ with $\sigma^h_h = \rho^b_h(\cdot|\operatorname{Supp}\sigma^b)$, there exists some r^a so that

$$\sum_{s^b} \pi^a(r^a, s^b)\rho^b_h(s^b) > \sum_{s^b} \pi^a(s^a, s^b)\rho^b_h(s^b).$$

Case 4.3 *Ann moves at h.*

Here, s^a

Since, inadmissible with respect to $[\Gamma^a_m \cap S^a(h)] \times [\Gamma^b_m \cap S^b(h)]$, it follows that, for each $\rho^b_h \in \mathcal{M}^+(\Gamma^b_m \cap S^b(h))$ with $\sigma^h_h = \rho^b_h$

, for each $\rho^b \in$

We divide the remainder of the proof into two cases. The first is where Ann moves at h and the second is where she does not.

Case 4.4 *Ann moves at h.*

Here, we can find information sets $1, \ldots, K$ so that $S^a(1), \ldots, S^a(K)$ form a partition of $S^a(h)$ and $S^b(k) = S^b(h)$ for all $k = 1, \ldots, K$. Suppose there is a strategy r^a such that $\pi^a(s^a, \sigma^b_h) = \pi^a(r^a, \sigma^b_h)$. Then r^a must specify the same move as s^a at h. (This is property (2) above.) Recall that s^a is admissible given $[\Gamma^a_m \cap S^a(k)] \times [\Gamma^b_m \cap S^b(k)]$. Since $S^b(k) = S^b(h)$, we have some measure $\rho^b_h \in \mathcal{M}^+(\Gamma^b_m \cap S^b(h))$ with $\pi^a(s^a, \rho^b_h) \geq \pi^a(q^a, \rho^b_h)$ for all $q^a \in \Gamma^a_m \cap S^a(k)$. Moreover, if $\pi^a(s^a, \sigma^b_h) = \pi^a(r^a, \sigma^b_h)$, then $\pi^a(s^a, \rho^b_h) \geq \pi^a(r^a, \rho^b_h)$. So, we can build a $\delta^b_h \in \mathcal{M}^+(\Gamma^b_m \cap S^b(h))$ by setting $\delta^b_h(s^b) = (1-\varepsilon)\sigma^b_h(s^b) + \varepsilon\rho^b_h(s^b)$, where $0 < \varepsilon < 1$. Then, there exists $\varepsilon > 0$ so that $\pi^a(s^a, \delta^b_h) \geq \pi^a(q^a, \delta^b_h)$ for all $q^a \in \Gamma^a_m \cap S^a(h)$. (Here, we use the fact that if $\pi^a(s^a, \sigma^b_h) = \pi^a(r^a, \sigma^b_h)$, then $\pi^a(s^a, \rho^b_h) \geq \pi^a(r^a, \rho^b_h)$.) This contradicts the fact that s^a is inadmissible with respect to $[\Gamma^a_m \cap S^a(h)] \times [\Gamma^b_m \cap S^b(h)]$.

Case II: Ann doesn't move at h. Here, we can find information sets $1, \ldots, K$ so that $S^b(1), \ldots, S^b(K)$ form a partition of $S^b(h)$ and $S^a(k) = S^a(h)$ for all $k = 1, \ldots, K$. Order the information sets so that $\Gamma^b_m \cap S^b(k) \neq \emptyset$ if $1 \leq k \leq J \leq K$ and $\Gamma^b_m \cap S^b(k) = \emptyset$ if $J < k \leq K$. For each $k = 1, \ldots, J$, there exists $\sigma^b_k \in \mathcal{M}^+(\Gamma^b_m \times S^b(k))$ such that $\pi^a(s^a, \sigma^b_k) \geq \pi^a(q^a, \sigma^b_k)$ for all $q^a \in \Gamma^a_m \cap S^a(h)$. Build $\sigma^b \in \mathcal{M}^+(\Gamma^b_m \cap S^b(h))$ so that $\sigma^b(s^b) = \frac{1}{J}\sigma^b_k(s^b)$ where $s^b \in S^b(k)$. Then

$$\pi^a(q^a, \sigma^b) = \frac{1}{J}\sum_{k=1}^{J} \pi^a(q^a, \sigma^b_k)$$

for any $q^a \in \Gamma^a_m \cap S^a(h)$. It follows (using σ^b) that s^a is admissible with respect to $[\Gamma^a_m \cap S^a(h)] \times [\Gamma^b_m \cap S^b(h)]$, again a contradiction. ■

5 Conditioning on Own vs. Others' Information Sets

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