Quantum Decision Theory

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Is decision theory invariant to the physical environment in which a decision is made?

This seems to be the conventional view.

Likewise, the old view in computer science was that the theory of computing could be developed without attention to the particular physical components (silicon, copper, etc.) from which computers are built.

“Computers might as well be made of green cheese.” *

The advent of quantum computing showed that the conventional view in computer science was wrong (algorithms: Deutsch-Jozsa 1992; Grover 1996; Shor 1997).

We will argue that the availability of quantum information resources means that the conventional view in decision theory is also wrong.

* Our thanks to Samson Abramsky, who attributes this aphorism to his Ph.D. advisor.
We first examine a classical baseline:

What happens when a decision maker (DM) has access to classical signals and can make his choices contingent on the realization of those signals?

We then ask:

Does giving the DM access to quantum, not just classical, signals, lead to an improvement in what he can achieve?

We can interpret the addition of signals --- classical or quantum --- to a decision problem in two ways:

i. Signals represent an extra resource which a DM might be able to employ.

ii. Signals are omnipresent in the environment, and this is simply the correct analysis of decision making.
The circular node belongs to Nature, and the square nodes belong to the DM.

This is a decision tree with **imperfect recall** (Kuhn 1950, 1953).

At information set $I_2$, the DM does not remember his previous choice (if any).

(Obviously, this is not yet a formal definition of perfect recall.)
Highest Expected Payoff

\[
\begin{array}{c|c|c}
\text{In-Left} & 2/3 \\
\text{In-Right} & 4/3 \\
\text{Out-Left} & 1/3 \\
\text{Out-Right} & 5/3 \\
\end{array}
\]
Perhaps, signals could carry information through the tree which the DM is unable to carry himself.

Signals might make up for a lack of memory.

Could they even be a constituent of memory?
Adding Signals

Heads

Left
Right
Heads

Out
Out
Left
Right
Tails

In
In
Left
Right
Heads

Tails

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Each possible path through the tree crosses certain information sets of the DM in a certain order.
Set $\delta = \eta = 1$. The expected payoff is $2 > \frac{5}{3}$!
But, the behavior of the second coin is affected by the toss of the first coin.

In this sense, information is carried between information sets, so that it is not surprising that there is an improvement.

The **No Signaling Condition**: Consider two tuples of information sets and the two associated signal probability measures. The marginals of these two measures --- with respect to common sub-tuples --- must agree.

(The terminology is from quantum mechanics, and can be a bit confusing in decision theory.)

In the previous example, the condition implies:

\[ \alpha = \varepsilon + \zeta \]

\[ \gamma = \varepsilon + \eta \]

which rules out \( \delta = \eta = 1 \).
Expected payoff from the strategy shown:

\[ \varepsilon \times (\frac{1}{3} \times 1 + \frac{2}{3} \times 0) + \zeta \times (\frac{1}{3} \times 1 + \frac{2}{3} \times 2) + \eta \times (\frac{1}{3} \times 2 + \frac{2}{3} \times 0) + \theta \times (\frac{1}{3} \times 0 + \frac{2}{3} \times 2) \]
Conjecture:

Fix a Kuhn tree. The highest expected payoff to a DM in an augmented tree with signals satisfying No Signaling is the same as that in the tree without signals.

This is false, as we shall see!

But we do have:

Proposition:

Fix a Kuhn tree. The highest expected payoff to a DM in an augmented tree with classical signals is the same as that in the tree without signals. Moreover, No Signaling will be satisfied.

Of course, we have to say what we mean by “classical”.
The Classicality Condition:

Let \( \{I_1, I_2, \ldots\} \) be the set of information sets for the DM. There is a probability measure \( \mu \) on the product, over all \( I_1, I_2, \ldots \), of the associated signal sets, such that: For each information tuple \( I_{i_1} I_{i_2} \ldots \) that arises in the tree, the probability measure \( \Pr_{I_{i_1} I_{i_2} \ldots} \) is obtained from \( \mu \) by marginalization.

In short, there is a joint state space!

Note: This condition is well-defined since, in a Kuhn tree, each path crosses a given information set at most once.
Proposition:
Classicality implies No Signaling.

Proof:
Immediate by the properties of marginals.

Proposition:
Fix a Kuhn tree. The highest expected payoff a DM can achieve with signals satisfying Classicality is the same as that without signals.

Proof:
Under Classicality, we can write the expected payoff to a strategy in the augmented tree as a convex combination of expected payoffs to strategies in the original tree.
Example #2

Assume $0 < m < M$.

The DM's expected payoff with classical signals is at most 0.

Proof: Analyze without signals and then appeal to the previous proposition.
**A Signal Structure**

\[
\Phi = \frac{2}{1 + \sqrt{5}} \quad \text{is the inverse of the Golden Ratio.}
\]

No Signaling is satisfied (use \( \Phi^2 + \Phi = 1 \)).
The expected payoff is \( \frac{1}{4} \times \Phi^5 \times m > 0! \)
This signal structure can be physically realized in a quantum-mechanical (QM) system (Hardy 1993).

The signal (Heads or Tails) at $I_1$ is the outcome obtained from performing a certain measurement on particle #1.

The signal (Heads or Tails) at $I_2$ is the outcome obtained from performing a different measurement on particle #1.

The signal (Heads or Tails) at $I_3$ is the outcome obtained from performing a certain measurement on particle #2.

The signal (Heads or Tails) at $I_4$ is the outcome obtained from performing a different measurement on particle #2.

The key is that the two particles are entangled in a particular way.
A Joint State Space?

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<tr>
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<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
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We know from our proposition about classicality that the signal structure cannot arise from a joint state space.

Let us also give a direct proof. We try:

\[ \mu(\omega^0) + \mu(\omega^1) + \mu(\omega^4) + \mu(\omega^9) = 0, \]

\[ \mu(\omega^0) + \mu(\omega^1) + \mu(\omega^8) + \mu(\omega^9) = 0, \]

\[ \mu(\omega^0) + \mu(\omega^2) + \mu(\omega^4) + \mu(\omega^6) = 0, \]

but then find this contradicts:

\[ \mu(\omega^0) + \mu(\omega^2) + \mu(\omega^8) + \mu(\omega^{10}) > 0! \]
QM says that the two measurements on particle #1 (resp. particle #2) cannot have jointly well-defined outcomes.

This is a statement of the *incompatibility* or *non-commutativity* of various observables in QM (most famously: position and momentum).

It is the physical reason why there is no joint state space.

So this is a striking case of how a ‘weakness’ (incompatibility) becomes a ‘strength’ (entanglement).

Related analyses:

- Is there is a **local hidden-variable model** that induces the empirical outcome probabilities (Bell 1964)?

- Is there a joint state space with a **signed probability measure** that induces the empirical outcome probabilities (Abramsky and Brandenburger, *New Journal of Physics*, 2011)? (Of course, all empirical probabilities must be non-negative.)
The dawn of quantum biology

The key to practical quantum computing and high-efficiency solar cells may lie in the messy green world outside the physics lab.

On the face of it, quantum effects and living organisms seem to occupy utterly different realms. The former are usually observed only on the nanometre scale, surrounded by harsh vacuum, ultra-low temperatures and a tightly controlled laboratory environment. The latter inhabit a macroscopic world that is warm, messy and anything but controlled. A quantum phenomenon such as 'coherence', in which the wave patterns of every part of a system stay in step, wouldn't last a microsecond in the tumultuous realm of the cell.

Or so everyone thought. But discoveries in recent years suggest that nature knows a few tricks that physicists don't: coherent quantum processes may well be ubiquitous in the natural world. Known or suspected examples range from the ability of birds to navigate using Earth's magnetic field to the inner workings of photosynthesis — the process by
<table>
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<th>Imperfect-recall Kuhn</th>
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<td>Quantum</td>
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</tr>
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<td>0</td>
<td>+</td>
<td>++</td>
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</tbody>
</table>
The formulation we have used so far:

```
Left  Right
   I
Left  Right
Heads  Tails
   SAME COIN
Heads  Tails
Left  Right
   Left  Right
   Left  Right
   Left  Right
```
A Second Formulation

EXCHANGEABLE COINS

Heads Tails
Left Right

Heads Tails
Left Right

Left Right

Left Right
The second formulation leaves unchanged our results so far.

But it can make a difference in non-Kuhn trees (Isbell 1957, Piccione and Rubinstein 1997):

![Diagram of a non-Kuhn tree with nodes labeled 0, 1, and 4.]
Adding i.i.d. Signals

The expected payoff is $5/4 > 1!$ (Isbell 1957)
Adding Exchangeable Signals

The expected payoff is $2 > \frac{5}{4}$!
<table>
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