A Purification Theorem for Perfect-Information Games*

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Kalmar [2, 1928-9] proved that Chess is strictly determined. Von Neumann-Morgenstern [5, 1944] proved the same for any finite two-person zero-sum perfect-information (PI) game. The latter result yields a minimax theorem for (finite) non-zero-sum PI games.[1] Fix a PI, and a player, Ann. Convert this game to a two-person zero-sum game between Ann and the other players (considered as one player), in which Ann gets the same payoffs as in the initial game. In the new game, the min max of Ann's payoffs is equal to the max min of Ann's payoffs. Since this statement involves only Ann's payoffs, it must also hold in the initial game.

In this note we first prove a generalization of this fact: The min max and max min of Ann's payoffs are still equal, when they are taken over subsets of strategies, provided the subsets for the players other than Ann are rectangular.

We then use this generalization to prove a purification result: Fix a PI game, a strategy s for Ann, and rectangular subsets of strategies for the players other than Ann. Suppose s is weakly dominated with respect to these subsets. That is, suppose there is a mixed strategy for Ann that gives her at least as high a payoff for each profile of pure strategies for the other players, taken from the given subsets, and a strictly higher payoff against at least one such profile. Then there is a pure strategy for Ann that weakly dominates s in the same sense.

Kalmar's [2, 1928-9] proof was forward looking. Another forward-looking argument is Kuhn's [3, 1950], [4, 1953] well-known proof that every finite PI game has a pure-strategy Nash equilibrium." Our arguments, too, are forward looking, proceeding by induction on the length of the tree.

1 Perfect-Information Games

An extensive-form PI game is defined as follows: Let $\{a, b\}$ be the set of players.[2] A tree is given by a finite set of nodes N, which are partially ordered by $\preceq$. The set N has a least element ϕ with respect to $\preceq$. That is, ϕ is the initial node. The set of terminal nodes is $Z = \{z \in N : z \preceq z' \rightarrow z = z'\}$.

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[1] Stated in, e.g., Ben Porath [1, 1997]. We are grateful to Elchanan Ben Porath for the following argument.

[2] This is for notational convenience only. Our arguments immediately extend to games with three or more players– provided the rectangularity conditions hold.

The set of non-terminal nodes is $X = N \backslash Z$. Information sets are given by sets $\{x\}$, where $x \in X$. Often, we will identify an information set $\{x\}$ with x. The set of choices available at node $x \in X$ is the set of immediate successors of x; denote this set by $C(x)$.[3] The mapping $\iota : X \to \{a, b\}$ specifies the player who moves at each information set. Let $X^a = \{x \in X : \iota(x) = a\}$, and define X^b analogously. Extensive-form payoff functions are given by $\Pi^a : Z \to \mathbb{R}$ and $\Pi^b : Z \to \mathbb{R}$.

A strategy for player a is a mapping from information sets into available choices. That is, the strategy set for a is $S^a = \times_{x \in X^a} C(x)$. Let $S = S^a \times S^b$. A strategy profile $s \in S$ determines a path through the tree. Let $\zeta : S \to Z$ where $\zeta(s) = z$ if and only if s reaches the terminal node z. Strategic-form payoff functions are given by $\pi^a = \Pi^a \circ \zeta$ and $\pi^b = \Pi^b \circ \zeta$.

2 A Generalized Minimax Theorem

Take $C(\phi) = \{1, .., K\}$. For each k, let $S_k^a = \times_{x \in X^a : k \preceq x} C(x)$. If $\iota(\phi) = a$, $S^a = C(\phi) \times \left(\times_{k=1}^K S_k^a\right)$. In this case, denote by $S^a(k) = \{k\} \times S_k^a$. If $\iota(\phi) \neq a$, $S^a = \times_{k=1}^K S_k^a$. Write $\text{proj}_{S_k^a} s^a = s_k^a$ if $s^a = (j, s_1^a, ..., s_k^a, ..., s_K^a)$.

Definition 2.1 *A subset Y^a of S^a is* ***rectangular*** *if it is a product set, i.e., if $Y^a = \times_{x \in X^a} Y^a(x)$ where, for each $x \in X^a$, $Y^a(x) \subseteq C(x)$ with $Y^a(x) \neq \emptyset$.*

Notice that if both Y^a and Y^b are rectangular subsets, then there is another PI game Γ' where Y^a and Y^b can be identified with strategy sets of Γ'. But, if rectangularity of one of these sets fail, there may be no such tree.[4] Thus, the following result is a generalization of the Minimax Theorem:

Theorem 2.1 *Fix a PI game. Let Y^a be a subset of S^a and Y^b be a rectangular subset of S^b. Then,*

$$\min_{s^b \in Y^b} \max_{s^a \in Y^a} \pi^a\left(s^a, s^b\right) = \max_{s^a \in Y^a} \min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right).$$

Proof. By induction on the length of the tree.

Length = 1: If $\iota(\phi) = a$ then $S^b = \{\emptyset\}$. So, certainly

$$\min_{s^b \in Y^b} \max_{s^a \in Y^a} \pi^a\left(s^a, s^b\right) = \max_{s^a \in Y^a} \pi^a\left(s^a, \emptyset\right) = \max_{s^a \in Y^a} \min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right).$$

If $\iota(\phi) = b$ then $S^a = \{\emptyset\}$. We then have

$$\min_{s^b \in Y^b} \max_{s^a \in Y^a} \pi^a\left(s^a, s^b\right) = \min_{s^b \in Y^b} \pi^a\left(\emptyset, s^b\right) = \max_{s^a \in Y^a} \min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right),$$

as desired.

Length $\geq$ 2: Assume the result is true for any tree of length λ or less. Fix a tree of length $\lambda + 1$.

First suppose $\iota(\phi) = a$. Since Y^b is rectangular, we can write $Y^b = \times_{k=1}^K Y_k^b$, where each Y_k^b is a rectangular subset of S_k^b.

Let $\left(\overline{s}_k^a, \overline{s}_k^b\right) \in \arg \max_{s^a \in Y^a \cap S^a(k)} \min_{s_k^b \in Y_k^b} \pi^a\left(s^a, s_k^b\right)$. Consider $\overline{s}^b \in S^b$ such that $\text{proj}_{S_k^b} \overline{s}^b = \overline{s}_k^b$ for each k. Since Y^b is rectangular, $\overline{s}^b \in Y^b$. Let $\left(\overline{s}_1^a, \overline{s}_1^b\right) \in Y^a \times Y_1^b$ be such that $\pi^a\left(\overline{s}_1^a, \overline{s}_1^b\right) \geq \pi^a\left(\overline{s}_k^a, \overline{s}_k^b\right)$ for each k with $Y^a \cap S^a(k) \neq \emptyset$.

[3] An immediate successor of x is a node $y \in N$ with $x \preceq y$ and $x \preceq y' \to y \preceq y'$.

[4] We thank Jeroen Swinkels for this point.

It follows that $\left(\overline{s}_1^a, \overline{s}^b\right) \in \arg\max_{s^a \in Y^a} \min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right)$: For any $s^a \in Y^a$, $s^b \in \arg\min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right)$ if and only if $\operatorname{proj}_{S_k^b} s^b \in \arg\min_{s_k^b \in Y_k^b} \pi^a\left(s^a, s_k^b\right)$ for k such that $s^a \in S^a(k)$. So, certainly, $\overline{s}^b \in \arg\min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right)$ for some $s^a \in Y^a$. Suppose there exists $\widehat{s}^b \in \arg\min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right)$ such that, for some $\widehat{s}^a \in Y^a$, $\pi^a(\widehat{s}^a, \widehat{s}^b) > \pi^a\left(\overline{s}_1^a, \overline{s}^b\right)$. Without loss of generality, take $(\widehat{s}^a, \operatorname{proj}_{S_k^b} \widehat{s}^b) \in \arg\max_{s^a \in Y^a \cap S^a(k)} \min_{s_k^b \in Y_k^b} \pi^a\left(s^a, s_k^b\right)$ for k with $\widehat{s}^a \in S^a(k)$. This contradicts the choice of $\left(\overline{s}_1^a, \overline{s}_1^b\right)$.

Similarly, for each k, let $(\underline{s}_k^a, \underline{s}_k^b) \in \arg\min_{s_k^b \in Y_k^b} \max_{s^a \in Y^a \cap S^a(k)} \pi^a\left(s^a, s_k^b\right)$. Let $\pi^a\left(\underline{s}^a, \underline{s}^b\right) := \min_{s^b \in Y^b} \max_{s^a \in Y^a} \pi^a\left(s^a, s^b\right)$. Define

$$\operatorname{MAX}(Y^a) = \{s^a \in Y^a : s^a \in \arg\max_{Y^a} \pi^a\left(\cdot, s^b\right) \text{ for some } s^b \in Y^b\}.$$

It will be shown that, for some j with $Y^a \cap S^a(j) \neq \emptyset$, $\pi^a(\underline{s}_j^a, \underline{s}_j^b) \geq \pi^a\left(\underline{s}^a, \underline{s}^b\right)$. To do so, it suffices to show that for some such j with $Y^a \cap S^a(j) \neq \emptyset$, $\underline{s}_j^a \in \operatorname{MAX}(Y^a)$. Then

$$\max_{Y^a} \pi^a\left(s^a, \underline{s}^b\right) \leq \pi^a(\underline{s}_j^a, s_j^b),$$

for all $s_j^b \in Y_j^b$. So, certainly

$$\max_{Y^a} \pi^a\left(s^a, \underline{s}^b\right) \leq \pi^a(\underline{s}_j^a, \underline{s}_j^b).$$

Pick j such that $\pi^a(\underline{s}_j^a, \underline{s}_j^b) \geq \pi^a\left(\underline{s}_k^a, \underline{s}_k^b\right)$ for each k with $Y^a \cap S^a(k) \neq \emptyset$. Suppose for each k with $Y^a \cap S^a(k) \neq \emptyset$, there exists s_k^b such that $\pi^a(\underline{s}_j^a, \underline{s}_j^b) \geq \max_{Y^a \cap S^a(k)} \pi^a\left(s^a, s_k^b\right) = \pi^a\left(\underline{s}_k^a, s_k^b\right)$. Pick $s^b \in S^b$ so that $\operatorname{proj}_{S_j^b} s^b = \underline{s}_j^b$, $\operatorname{proj}_{S_k^b} s^b = s_k^b$ for k with $Y^a \cap S^a(k) \neq \emptyset$, and $\operatorname{proj}_{S_k^b} s^b \in Y_k$ otherwise. Since Y^b is rectangular, $s^b \in Y^b$. Then $\underline{s}_j^a \in \operatorname{MAX}(Y^a)$. So, if $\underline{s}_j^a \notin \operatorname{MAX}(Y^a)$, there exists k with $Y^a \cap S^a(k) \neq \emptyset$ such that $\pi^a\left(\underline{s}_k^a, s_k^b\right) > \pi^a(\underline{s}_j^a, \underline{s}_j^b)$, for all $s_k^b \in Y_k^b$. Certainly, for this k, $\pi^a\left(\underline{s}_k^a, \underline{s}_k^b\right) > \pi^a(\underline{s}_j^a, \underline{s}_j^b)$, a contradiction.

With this $\pi^a\left(\overline{s}_1^a, \overline{s}^b\right) = \pi^a\left(\underline{s}^a, \underline{s}^b\right)$. Indeed, suppose not. Then

$$\begin{aligned}
\pi^a\left(\overline{s}_1^a, \overline{s}^b\right) &< \pi^a\left(\underline{s}^a, \underline{s}^b\right) \\
&\leq \pi^a(\underline{s}_j^a, \underline{s}_j^b) = \pi^a\left(\overline{s}_j^a, \overline{s}_j^b\right) \\
&\leq \pi^a\left(\overline{s}_1^a, \overline{s}_1^b\right) = \pi^a\left(\overline{s}_1^a, \overline{s}^b\right),
\end{aligned}$$

where the first line comes from $\max\min \pi^a(\cdot, \cdot) \leq \min\max \pi^a(\cdot, \cdot)$, the second line comes from the choice of $(\underline{s}_j^a, \underline{s}_j^b)$ and the induction hypothesis, and the third line comes from the choice of $\left(\overline{s}_1^a, \overline{s}_1^b\right)$. But this is a contradiction.

Now suppose $\iota(\phi) \neq a$. For each k, let $\pi^a\left(\overline{r}_k^a, \overline{r}_k^b\right) = \max_{s_k^a \in Y_k^a} \min_{s^b \in Y^b \cap S^b(k)} \pi^a\left(s^a, s_k^b\right)$. Further, let $\pi^a\left(\overline{r}^a, \overline{r}^b\right) = \max_{s^a \in Y^a} \min_{s^b \in Y^b} \pi^a\left(s^a, s^b\right)$. Define

$$\operatorname{MIN}\left(Y^b\right) = \{s^b \in Y^b : s^b \in \arg\min_{Y^b} \pi^a\left(s^a, \cdot\right) \text{ for some } s^a \in Y^a\}.$$

It will be shown that, for some 1 with $Y^b \cap S^b(1) \neq \emptyset$, $\pi^a\left(\overline{r}_1^a, \overline{r}_1^b\right) \leq \pi^a\left(\overline{r}^a, \overline{r}^b\right)$. To do so, it suffices to show that for some such 1 with $Y^b \cap S^b(1) \neq \emptyset$, $\overline{r}_1^b \in \operatorname{MIN}\left(Y^b\right)$. Then

$$\min_{Y^b} \pi^a\left(\overline{r}^a, s^b\right) \geq \pi^a\left(r_1^a, \overline{r}_1^b\right),$$

for all $r_1^a \in Y_1^a$. So, certainly

$$\min_{Y^b} \pi^a \left(\overline{r}^a, s^b\right) \geq \pi^a \left(\overline{r}_1^a, \overline{r}_1^b\right).$$

Pick 1 such that $\pi^a \left(\overline{r}_1^a, \overline{r}_1^b\right) \leq \pi^a \left(\overline{r}_k^a, \overline{r}_k^b\right)$ for each $k : Y^b \cap S^b(k) \neq \emptyset$. Suppose for each k with $Y^b \cap S^b(k) \neq \emptyset$, there exists r_k^a such that $\pi^a \left(\overline{r}_1^a, \overline{r}_1^b\right) \leq \min_{Y^b \cap S^b(k)} \pi^a \left(r_k^a, s^b\right) = \pi^a \left(r_k^a, \overline{r}_k^b\right)$. Pick $r^a \in S^a$ so that $\text{proj}_{S_1^a} r^a = \overline{r}_1^a$, $\text{proj}_{S_k^a} r^a = r_k^a$ for k with $Y^b \cap S^b(k) \neq \emptyset$, and $\text{proj}_{S_k^a} r^a \in Y_k^a$ otherwise. Since Y^a is rectangular, $r^a \in Y$. Then $\overline{r}_1^b \in \text{MIN}\left(Y^b\right)$. So, if $\overline{r}_1^b \notin \text{MIN}\left(Y^b\right)$, there exists k with $Y^b \cap S^b(k) \neq \emptyset$ such that $\pi^a \left(r_k^a, \overline{r}_k^b\right) < \pi^a \left(\overline{r}_1^a, \overline{r}_1^b\right)$, for all $r_k^a \in Y_k^a$. Certainly, for this k, $\pi^a \left(\overline{r}_k^a, \overline{r}_k^b\right) < \pi^a \left(\overline{r}_1^a, \overline{r}_1^b\right)$, a contradiction.

Let $\pi^a \left(\underline{r}_k^a, \underline{r}_k^b\right) = \min_{s^b \in Y^b \cap S^b(k)} \max_{s_k^a \in Y_k^a} \pi^a \left(s_k^a, s^b\right)$. Define $\underline{r}^a \in S^a$ so that $\text{proj}_{S_k^a} \underline{r}^a = \underline{r}_k^a$; note that $\underline{r}^a \in Y$, as Y is rectangular. Pick $\left(\underline{r}_j^a, \underline{r}_j^b\right)$ so that $\pi^a \left(\underline{r}_j^a, \underline{r}_j^b\right) \leq \pi^a \left(\underline{r}_k^a, \underline{r}_k^b\right)$ for each k with $Y^b \cap S^b(k) \neq \emptyset$.

Note, $(\underline{r}^a, \underline{r}_j^b) \in \arg \min\limits_{s^b \in Y^b} \max\limits_{s^a \in Y^a} \pi^a \left(s^a, s^b\right)$: For each $s^b \in Y^b \cap S^b(k)$, $s^a \in \arg\max_{Y^a} \pi^a \left(s^a, s^b\right)$ if and only if $\text{proj}_{S_k^a} s^a \in \arg\max_{Y_k^a} \pi^a \left(s_k^a, s^b\right)$. So, certainly, $\underline{r}^a \in \arg\max_{Y^a} \pi^a \left(s^a, s^b\right)$ for some $s^b \in Y^b$. Suppose there exists $\widehat{r}^b \in Y^b \cap S^b(k)$ such that $\pi^b(\widehat{r}^a, \widehat{r}^b) < \pi^b(\underline{r}^a, \underline{r}_j^b)$ for some $\widehat{r}^a \in \arg\max_{Y^a} \pi^a \left(s^a, s^b\right)$. Without loss of generality, take $(\widehat{r}^a, \widehat{r}^b)$ so that $\pi^a(\widehat{r}^a, \widehat{r}^b) = \min_{r^b \in Y^b \cap S^b(k)} \max_{s_k^a \in Y_k^a} \pi^a \left(s_k^a, s^b\right)$. But this contradicts $\pi^a(\underline{r}_j^a, \underline{r}_j^b) \leq \pi^a \left(\underline{r}_k^a, \underline{r}_k^b\right)$ for each k with $Y^b \cap S^b(k) \neq \emptyset$.

It follows that $\pi^a \left(\overline{r}_1^a, \overline{r}^b\right) = \pi^a(\underline{r}_j^a, \underline{r}^b)$, since

$$\begin{aligned}
\pi^a(\underline{r}^a, \underline{r}_j^b) &= \pi^a(\underline{r}_j^a, \underline{r}_j^b) \\
&\leq \pi^a \left(\underline{r}_1^a, \underline{r}_1^b\right) \\
&= \pi^a \left(\overline{r}_1^a, \overline{r}_1^b\right) \\
&\leq \pi^a \left(\overline{r}^a, \overline{r}^b\right) \leq \pi^a(\underline{r}^a, \underline{r}_j^b),
\end{aligned}$$

where the second line comes from the choice of $(\underline{r}_j^a, \underline{r}_j^b)$, the third line comes from the induction hypothesis, and the fourth line comes from the choice of $\left(\overline{r}_1^a, \overline{r}_1^b\right)$ and the fact that $\max\min \pi^a(\cdot,\cdot) \leq \min\max \pi^a(\cdot,\cdot)$. ■

3 A Purification Theorem

We need a preliminary result: If a strategy for Ann is inadmissible with respect to a rectangular subset of strategies for Bob, then Ann's strategy must be weakly dominated by a mixture that reaches a single subtree k.

Some notation: For a finite set X, let $\mathcal{M}(X)$ denote set of all probability measures on X. Then:

Lemma 3.1 *Fix a PI game with $\iota(\phi) = a$. If s^a is inadmissible with respect to a rectangular subset Y^b of S^b, then there exists k and $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(S^a(k)) = 1$, such that*

$$\begin{aligned}
&\pi^a \left(\sigma^a, s^b\right) \geq \pi^a \left(s^a, s^b\right) \quad \textit{for all } s^b \in Y^b, \\
&\pi^a \left(\sigma^a, s^b\right) > \pi^a \left(s^a, s^b\right) \quad \textit{for some } s^b \in Y^b.
\end{aligned}$$

Proof. Let $\widehat{\sigma}^a \in \mathcal{M}(S^a)$ be such that

$$\begin{aligned}
&\pi^a \left(\widehat{\sigma}^a, s^b\right) \geq \pi^a \left(s^a, s^b\right) \quad \text{for all } s^b \in Y^b, \\
&\pi^a \left(\widehat{\sigma}^a, s^b\right) > \pi^a \left(s^a, s^b\right) \quad \text{for some } s^b \in Y^b.
\end{aligned}$$

Let $k = 1, .., \overline{K} \leq K$ be such that $\widehat{\sigma}^a(r^a) > 0$ for some $r^a \in S^a(k)$. For each $k = 1, .., \overline{K}$, define

$$\widehat{\sigma}^a_k(r^a) = \frac{\widehat{\sigma}^a(r^a)}{\sum_{q^a \in S^a(k)} \widehat{\sigma}^a(q^a)}.$$

It is readily verified that $\widehat{\sigma}^a_k(S^a(k)) = 1$. Also, for each $s^b \in Y^b$,

$$\pi^a\left(\widehat{\sigma}^a, s^b\right) = \sum_{k=1}^{\overline{K}}[(\sum_{q^a \in S^a(k)} \widehat{\sigma}^a(q^a)) \times \pi^a\left(\widehat{\sigma}^a_k, s^b\right)].$$

Suppose, for each $k = 1, .., \overline{K}$, $\widehat{\sigma}^a_k$ does not weakly dominate s^a with respect to Y^b. Then, for each such k, either: (i) $\pi^a\left(\widehat{\sigma}^a_k, s^b\right) = \pi^a\left(s^a, s^b\right)$ for each $s^b \in Y^b$; or (ii) there exists $r^b \in Y^b$ with $\pi^a\left(s^a, r^b\right) > \pi^a\left(\widehat{\sigma}^a_k, r^b\right)$. Notice that if (i) holds for all $k = 1, .., \overline{K}$, then for all $s^b \in Y^b$,

$$\pi^a\left(\widehat{\sigma}^a, s^b\right) = \sum_{k=1}^{\overline{K}}[(\sum_{q^a \in S^a(k)} \widehat{\sigma}^a(q^a)) \times \pi^a\left(s^a, s^b\right)] = \pi^a\left(s^a, s^b\right),$$

a contradiction. So, (ii) must hold for some such k.

For each $k = 1, ..., \overline{K}$ satisfying (ii), let $r^b_k = \mathrm{proj}_{S^b(k)}\, r^b$ and, if $s^a \in S^a(j)$ for $j \neq k$, let $r^b_j = \mathrm{proj}_{S^b(j)}\, r^b$. Set $r^b = \left(r^b_1, ..., r^b_K\right)$ where r^b_k is as above, if defined, and is otherwise an arbitrary element of Y_k. Since Y^b is rectangular, $r^b \in Y^b$. But

$$\pi^a\left(\widehat{\sigma}^a, r^b\right) = \sum_{k=1}^{\overline{K}}[(\sum_{q^a \in S^a(k)} \widehat{\sigma}^a(q^a)) \times \pi^a\left(\widehat{\sigma}^a_k, r^b\right)] < \pi^a\left(s^a, r^b\right),$$

where the inequality comes from the fact that (ii) holds for some $k = 1, ..., \overline{K}$. ■

We now come to the purification result:

Theorem 3.1 *Fix a PI game. If s^a is inadmissible with respect to a rectangular subset Y^b of S^b, then there exists $r^a \in S^a$ such that*

$$\begin{array}{ll} \pi^a\left(r^a, s^b\right) \geq \pi^a\left(s^a, s^b\right) & \textit{for all } s^b \in Y^b, \\ \pi^a\left(r^a, s^b\right) > \pi^a\left(s^a, s^b\right) & \textit{for some } s^b \in Y^b. \end{array}$$

Proof. By induction on the length of the tree.

Length = 1: Without loss of generality, let $\iota(\phi) = a$. Here $S^b = \{\emptyset\}$, so each s^b is admissible. Moreover, if there exists $\sigma^a \in \mathcal{M}(S^a)$ such that $\pi^a(\sigma^a, \emptyset) > \pi^a(s^a, \emptyset)$ then certainly there exists $r^a \in \mathrm{Supp}\, \sigma^a$ such that $\pi^a(r^a, \emptyset) > \pi^a(s^a, \emptyset)$, as required.

Length ≥ 2: Assume the result is true for any tree of length λ or less. Fix a tree of length $\lambda + 1$.

First, suppose $\iota(\phi) = a$. Fix $s^a \in S^a(1)$ and suppose s^a is inadmissible with respect to a rectangular Y^b. Then, by Lemma 3.1, there exists k with $\sigma^a(S^a(k)) = 1$ such that, for all $s^b \in Y^b$, $\pi^a\left(\sigma^a, s^b\right) \geq \pi^a\left(s^a, s^b\right)$, with strict inequality for some $s^b \in Y^b$. Without loss of generality, pick σ^a so that $r^a \in \mathrm{Supp}\, \sigma^a$ implies that there exists $s^b \in Y^b$ with $\pi^a\left(s^a, s^b\right) \neq \pi^a\left(r^a, s^b\right)$.

Suppose first that $\mathrm{Supp}\, \sigma^a \subseteq S^a(1)$. Then by the induction hypothesis, there exists $r^a \in S^a(1)$ such that

$$\begin{array}{ll} \pi^a\left(r^a, s^b_1\right) \geq \pi^a\left(s^a, s^b_1\right) & \forall s^b_1 \in Y^b_1 \\ \pi^a\left(r^a, s^b_1\right) > \pi^a\left(s^a, s^b_1\right) & \text{for some } s^b_1 \in Y^b_1 \end{array};$$

the result is then immediate. So, let $\operatorname{Supp}\sigma^a \subseteq S^a(k)$, for $k \neq 1$.

Pick $r_1^b \in \arg\max_{s_1^b \in Y_1} \pi^a(s^a, s_1^b)$ and let $Y_0^b = \{s^b \in Y^b : \operatorname{proj}_{Y_1^b} s^b = r_1^b\}$. Let $Y_0^a := \operatorname{Supp}\sigma$. Let $\pi^a(\overline{s}^a, \overline{s}^b) = \max_{Y_0^a}\min_{Y_0^b}\pi^a(\cdot,\cdot)$ and $\pi^a(\underline{s}^a, \underline{s}^b) = \min_{Y_0^b}\max_{Y_0^a}\pi^a(\cdot,\cdot)$. Then, for all $s^b \in Y^b$,

$$\begin{aligned}
\pi^a(s^a, s^b) &\leq \pi^a(s^a, \underline{s}^b) \\
&\leq \pi^a(\underline{s}^a, \underline{s}^b) \\
&= \pi^a(\overline{s}^a, \overline{s}^b) \\
&\leq \pi^a(\overline{s}^a, s^b),
\end{aligned}$$

where the first line comes from the choice of r_1^b, the second line from the definition of $\min\max$, the third line from Theorem 2.1, and the last line from the definition of $\max\min$. By choice of σ^a, there exists $s^b \in Y^b$, with $\pi^a(\overline{s}^a, s^b) > \pi^a(s^a, s^b)$, establishing the desired result.

Now suppose $\iota(\phi) \neq a$. Let $\sigma^a \in \mathcal{M}(S^a)$ be such that, for all $s^b \in Y^b$, $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$, with strict inequality for some $s^b \in Y$. Write $s^a = (s_1^a, .., s_K^a)$. For each k with $Y^b \cap S^b(k) \neq \emptyset$, define

$$\sigma_k^a(r_k^a) = \sum_{r^a \in S^a : \operatorname{proj}_{S_k^a} r^a = r_k^a} \sigma^a(r^a).$$

It is readily verified that $\sigma_k^a(S^a(k)) = 1$. For each k with $Y^b \cap S^b(k) \neq \emptyset$, $\pi^a(\sigma_k^a, s^b) \geq \pi^a(s_k^a, s^b)$ for all $s^b \in Y^b \cap S^b(k)$. Moreover, there exists k, with $s^b \in Y^b \cap S^b(k)$, such that $\pi^a(\sigma_k^a, s^b) > \pi^a(s_k^a, s^b)$.

By the induction hypothesis, for each such k, there exists $r_k^a \in S^a(k)$ such that $\pi^a(r_k^a, s^b) \geq \pi^a(s_k^a, s^b)$ for all $s^b \in Y^b \cap S^b(k)$, with strict inequality for some $s^b \in Y^b \cap S^b(k)$. For all other k, set $r_k^a = s_k^a$. Set $r^a = (r_1^a, ..., r_K^a)$. It is readily verified that r^a satisfies the desired properties. ■

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