Intrinsic Correlation in Games: Online Appendix*

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These appendices expand on Appendix I in the main text. Appendix J treats payoff uncertainty and Appendix K treats dummy players.

Appendix J Payoff Uncertainty

Here we introduce uncertainty over the payoff functions. Of course, if we introduce a 'substantial' amount of uncertainty about the payoff functions, there is no reason to expect that the correlated rationalizable strategies (of the original game) will be characterized by our conditions (in the new game). The idea will be to introduce a small amount of uncertainty–specifically, we assume that the given game is common $(1-\varepsilon)$-belief (Monderer-Samet [2, 1989]).

We show: *Fix any game G. Every correlated rationalizable strategy in G is consistent with CI, SUFF, RCBR, and common $(1-\varepsilon)$-belief of G.* But for any $\varepsilon > 0$, correlated rationalizability does not characterize these conditions. We show: *Fix $\varepsilon > 0$. There is a game G, an associated type structure satisfying CI and SUFF, and a state at which the game is indeed G, there is common $(1-\varepsilon)$-belief of G, RCBR holds, but the strategies played aren't correlated rationalizable in G.*

To state these results, we need to extend the definitions in the main text, to allow uncertainty over payoff functions.

Fix an n-**player strategic game form** $\langle S^1, .., S^n \rangle$, and a player i. An i-**payoff function** is a map $\pi^i : S \to \mathbb{R}$.

Definition J1 *An $(S^1, P^1, \ldots, S^n, P^n)$-**based type structure** is a structure*

$$\Phi = \langle S^1, \ldots, S^n; P^1, \ldots, P^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n \rangle,$$

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where each P^i is a Polish space of i-payoff functions, each T^i is a Polish space, and each $\lambda^i : T^i \to \mathcal{M}(S^{-i} \times P^{-i} \times T^{-i})$ is continuous.

Note that an $(S^1, P^1, \ldots, S^n, P^n)$-based type structure induces hierarchies of beliefs about the structure of the game (the payoff functions) as well as the strategies played in the game. To see this, extend the definition of the sets Y^i_m in the main text by setting $Y^i_1 = S^{-i} \times P^{-i}$ and defining inductively $Y^i_{m+1} = Y^i_m \times \prod_{j \neq i} \mathcal{M}(Y^j_m)$. Define continuous maps $\rho^i_m : S^{-i} \times P^{-i} \times T^{-i} \to Y^i_m$ by

$$\begin{aligned} \rho^i_1\left(s^{-i}, \pi^{-i}, t^{-i}\right) &= \left(s^{-i}, \pi^{-i}\right), \\ \rho^i_{m+1}\left(s^{-i}, \pi^{-i}, t^{-i}\right) &= (\rho^i_m\left(s^{-i}, \pi^{-i}, t^{-i}\right), (\delta^j_m\left(t^j\right))_{j \neq i}), \end{aligned}$$

where $\delta^j_m = \underline{\rho}^j_m \circ \lambda^j$.[1] Let $\delta^i : T^i \to \prod_{m=1}^{\infty} \mathcal{M}\left(Y^i_m\right)$ where $\delta^i\left(t^i\right) = \left(\delta^i_1\left(t^i\right), \delta^i_2\left(t^i\right), \ldots\right)$. Also, let $\delta^{-i} : T^{-i} \to \prod_{j \neq i} \prod_{m=1}^{\infty} \mathcal{M}(Y^j_m)$ be the product of the maps δ^j for each $j \neq i$.

Next, we extend the definitions of CI and SUFF. Fix a type structure Φ. For each player $i = 1, \ldots, n$ and each $j \neq i$, define random variables $\overrightarrow{s}^j_i$ and $\overrightarrow{t}^j_i$ on $S^{-i} \times P^{-i} \times T^{-i}$ by $\overrightarrow{s}^j_i = \text{proj}_{S^j}$ and $\overrightarrow{t}^j_i = \text{proj}_{T^j}$. Let $\overrightarrow{t}_i$ be the random variable on $S^{-i} \times P^{-i} \times T^{-i}$ given by $\overrightarrow{t}_i = \text{proj}_{T^{-i}}$. Define $\eta^j_i : S^{-i} \times P^{-i} \times T^{-i} \to P^j \times \prod_{m=1}^{\infty} \mathcal{M}(Y^j_m)$ and $\eta^{-i} : S^{-i} \times P^{-i} \times T^{-i} \to P^{-i} \times \prod_{j \neq i} \prod_{m=1}^{\infty} \mathcal{M}(Y^j_m)$ by

$$\begin{aligned} \eta^j_i\left(s^{-i}, \pi^{-i}, t^{-i}\right) &= (\pi^j, (\delta^j \circ \overrightarrow{t}^j_i)(s^{-i}, \pi^{-i}, t^{-i})), \\ \eta^{-i}\left(s^{-i}, \pi^{-i}, t^{-i}\right) &= (\pi^{-i}, (\delta^{-i} \circ \overrightarrow{t}_i)(s^{-i}, \pi^{-i}, t^{-i})). \end{aligned}$$

Note η^j_i and η^{-i} are products of random variables and so are random variables themselves.

Definition J2 *The random variables $\overrightarrow{s}^1_i, \ldots, \overrightarrow{s}^{i-1}_i, \overrightarrow{s}^{i+1}_i, \ldots, \overrightarrow{s}^n_i$ are $\lambda^i(t^i)$-**conditionally independent given the random variable** η^{-i} if, for all $j \neq i$ and $E^j \in \sigma(\overrightarrow{s}^j_i)$,*

$$\lambda^i\left(t^i\right)(\textstyle\bigcap_{j \neq i} E^j || \sigma(\eta^{-i})) = \prod_{j \neq i} \lambda^i\left(t^i\right)\left(E^j || \sigma(\eta^{-i})\right) \quad a.s.$$

*Say the type t^i satisfies **conditional independence (CI)** if $\overrightarrow{s}^1_i, \ldots, \overrightarrow{s}^{i-1}_i, \overrightarrow{s}^{i+1}_i, \ldots, \overrightarrow{s}^n_i$ are $\lambda^i(t^i)$-conditionally independent given η^{-i}.*

Definition J3 *The random variable η^j_i is $\lambda^i(t^i)$-**sufficient for the random variable** $\overrightarrow{s}^j_i$ if, for each $j \neq i$ and $E^j \in \sigma(\overrightarrow{s}^j_i)$,*

$$\lambda^i\left(t^i\right)\left(E^j || \sigma(\eta^{-i})\right) = \lambda^i\left(t^i\right)(E^j || \sigma(\eta^j_i)) \quad a.s.$$

*Say the type t^i satisfies **sufficiency (SUFF)** if, for each $j \neq i$, η^j_i is $\lambda^i(t^i)$-sufficient for $\overrightarrow{s}^j_i$.*

The next step is to extend the definition of RCBR.

[1] The proof that the maps ρ^i_m are continuous parallels the proof of Proposition B1.

Definition J4 *Say* $\left(s^i, \pi^i, t^i\right) \in S^i \times P^i \times T^i$ *is **rational** if*

$$\sum_{s^{-i} \in S^{-i}} \pi^i\left(s^i, s^{-i}\right) \operatorname{marg}_{S^{-i}} \lambda^i\left(t^i\right)\left(s^{-i}\right) \geq \sum_{s^{-i} \in S^{-i}} \pi^i\left(r^i, s^{-i}\right) \operatorname{marg}_{S^{-i}} \lambda^i\left(t^i\right)\left(s^{-i}\right)$$

for every $r^i \in S^i$. *Let* R_1^i *be the set of all rational triples* $\left(s^i, \pi^i, t^i\right)$.

Note that, in contrast with Definition 10.1 in the main text, rationality is now a property of a strategy-payoff function-type triple, not just a strategy-type pair.

Definition J5 *Say* $E \subseteq S^{-i} \times P^{-i} \times T^{-i}$ *is **believed under*** $\lambda^i\left(t^i\right)$ *if* E *is Borel and* $\lambda^i\left(t^i\right)(E) = 1$.

Define $B^i(E)$ as before, and for $m > 1$, define R_m^i inductively by

$$R_{m+1}^i = R_m^i \cap \left[S^i \times P^i \times B^i\left(R_m^{-i}\right)\right].$$

Definition J6 *If* $\left(s^1, \pi^1, t^1, \ldots, s^n, \pi^n, t^n\right) \in R_{m+1}$, *say there is **rationality and*** m***th-order belief of rationality (R****m****BR)** at this state. If* $\left(s^1, \pi^1, t^1, \ldots, s^n, \pi^n, t^n\right) \in \bigcap_{m=1}^{\infty} R_m$, *say there is **rationality and common belief of rationality (RCBR)** at this state.*

Finally, we want to formalize common $(1-\varepsilon)$-belief of the game. For each i, fix $E^i \subseteq S^i \times P^i \times T^i$. Let

$$B^i(E^{-i}; \varepsilon) = \{t^i \in T^i : \lambda^i\left(t^i\right)\left(E^{-i}\right) \geq 1 - \varepsilon\}.$$

Fix some $\varepsilon > 0$. Set $E_1^i = E^i$, and for $m \geq 1$ define inductively

$$E_{m+1}^i = E_m^i \cap [S^i \times P^i \times B^i(E_m^{-i}; \varepsilon)].$$

Definition J7 (Monderer-Samet [2, 1989]) *If* $\left(s^1, \pi^1, t^1, \ldots, s^n, \pi^n, t^n\right) \in \bigcap_{m=1}^{\infty} E_m$, *say that* E ***holds and there is common*** $(1-\varepsilon)$***-belief of*** E *at this state.*

We can now state and prove our first result on payoff uncertainty.

Proposition J1 *Fix* $\varepsilon > 0$. *Fix also a game* $G = \left\langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n\right\rangle$, *and a BRS* $\prod_{i=1}^n Q^i$ *of* G. *Then there is a type structure* $\langle S^1, \ldots, S^n; P^1, \ldots, P^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n\rangle$ *such that for each profile* $\left(s^1, \ldots, s^n\right) \in \prod_{i=1}^n Q^i$, *there is a state* $\left(s^1, \pi^1, t^1, \ldots, s^n, \pi^n, t^n\right)$ *where:*

(i) RCBR holds;

(ii) E *holds and there is common* $(1-\varepsilon)$*-belief of* E, *where* $E^i = S^i \times \{\pi^i\} \times T^i$;

(iii) the types $t^1, \ldots, t^n$ *satisfy CI and SUFF.*

In words, Proposition J1 says that any strategy profile in a BRS of the game G–in particular then, any correlated rationalizable profile in G–can be played under the conditions of CI, SUFF, and RCBR, when G is indeed the game and there is common $(1-\varepsilon)$-belief of G. In this sense, we rescue the converse direction (part (ii)) of Proposition 10.1, with the help of payoff uncertainty.

To prove Proposition J1, we start by constructing, for each $s^i \in Q^i$, a total of $\left|Q^{-i}\right|$ i-payoff functions as follows. For each i, begin with an injective map $s^{-i} \longmapsto a\left(s^{-i}\right) > 0$ such that, for each $s^i \in Q^i$, there is some $r^{-i} \in Q^{-i}$ with $\pi^i\left(s^i, r^{-i}\right) \neq a\left(r^{-i}\right)$. Let $\gamma^i\left(s^i, s^{-i}\right) : S \to \mathbb{R}$ be an i-payoff function where

$$\gamma^i\left(s^i, s^{-i}\right)\left(r^i, r^{-i}\right) = \begin{cases} a\left(s^{-i}\right) & \text{if } r^i = s^i, \\ 0 & \text{otherwise.} \end{cases}$$

Note, if $s^i \neq r^i$ then, for all $s^{-i}, r^{-i} \in Q^{-i}$, $\gamma^i\left(s^i, s^{-i}\right) \neq \gamma^i\left(r^i, r^{-i}\right)$. By construction, this implies $\gamma^i\left(s^i, s^{-i}\right) \neq \gamma^i\left(r^i, r^{-i}\right)$ for any $\left(s^i, s^{-i}\right) \neq \left(r^i, r^{-i}\right)$.

Fix $s^{-i} \in S^{-i}$. Write $\Gamma\left(s^i; s^{-i}\right)$ for the profile of payoff functions where

$$\Gamma\left(s^i; s^{-i}\right) =$$
$$\left(\gamma^1\left(s^1, s^{-i-1}, s^i\right), \ldots, \gamma^{i-1}\left(s^{i-1}, s^{-i-(i-1)}, s^i\right), \gamma^{i+1}\left(s^{i+1}, s^{-i-(i+1)}, s^i\right), \ldots, \gamma^n\left(s^n, s^{-i-n}, s^i\right)\right).$$

Note, for any $s^{-i} \neq r^{-i}$ and any $j \neq i$, $\gamma^j\left(s^j, s^{-i-j}, s^i\right) \neq \gamma^j\left(r^j, r^{-i-j}, s^i\right)$. So, whenever $s^{-i} \neq r^{-i}$, $\Gamma\left(s^i; s^{-i}\right) \neq \Gamma\left(s^i; r^{-i}\right)$.

We now build a type structure Φ as follows. Set $T^i = Q^i$. For each i, let P^i be the set of all $\gamma^i\left(s^i, s^{-i}\right)$ plus π^i. Then each P^i is finite with cardinality $\left|Q^i\right| \times \left|Q^{-i}\right| + 1$. Set $T^i = Q^i$. Define $\lambda^i : T^i \to \mathcal{M}\left(S^{-i} \times P^{-i} \times T^{-i}\right)$ so that, for each $s^i \in T^i = Q^i$,

$$\lambda^i\left(s^i\right)\left(s^{-i}, \tilde{\pi}^{-i}, t^{-i}\right) = \begin{cases} (1-\varepsilon) \times \mu\left(s^i\right)\left(s^{-i}\right) & \text{if } \tilde{\pi}^{-i} = \pi^{-i} \text{ and } t^{-i} = s^{-i}, \\ \varepsilon \times \mu\left(s^i\right)\left(s^{-i}\right) & \text{if } \tilde{\pi}^{-i} = \Gamma\left(s^i; s^{-i}\right) \text{ and } t^{-i} = s^{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

where $\mu\left(s^i\right) \in \mathcal{M}\left(S^{-i}\right)$, with $\mu\left(s^i\right)\left(Q^{-i}\right) = 1$, and s^i is optimal under $\mu\left(s^i\right)$ under the payoff function π^i.

We show that each of the properties of the theorem is satisfied.

Proof of RCBR. We show by induction that for all m, the following holds: For all i and all $\left(s^i, s^{-i}\right) \in S$, $\left(s^i, \pi^i, s^i\right), \left(s^i, \gamma^i\left(s^i, s^{-i}\right), s^i\right) \in R^i_m$.

Begin with $m = 1$. We have that $\text{marg}_{S^{-i}} \lambda^i\left(s^i\right) = \mu\left(s^i\right)$, so that $\left(s^i, \pi^i, s^i\right) \in R^i_1$. By construction, s^i is strongly dominant under the i-payoff function $\gamma^i\left(s^i, s^{-i}\right)$, so that $\left(s^i, \gamma^i\left(s^i, s^{-i}\right), s^i\right) \in R^i_1$.

Next assume the result holds for $m \geq 1$. It is then immediate from the definition of $\lambda^i\left(s^i\right)$ and the induction hypothesis that $\lambda^i\left(s^i\right)\left(R^{-i}_m\right) = 1$, as required. ■

Proof of Common $(1-\varepsilon)$ **Belief.** We show by induction that $E^i_m = S^i \times \left\{\pi^i\right\} \times T^i$ and $B^i\left(E^{-i}_m; \varepsilon\right) = T^i$ for all m. For $m = 1$, this is immediate from the construction. Assume the result holds for m. Then

$$\begin{aligned} E^i_{m+1} &= E^i_m \cap [S^i \times P^i \times B^i(E^{-i}_m;\varepsilon)] \\ &= E^i_m \cap [S^i \times P^i \times T^i] \\ &= S^i \times \{\pi^i\} \times T^i, \end{aligned}$$

where the second and third lines follow from the induction hypothesis. Next note that for each $s^i \in T^i$,

$$\begin{aligned} \lambda^i\left(s^i\right)\left(E^{-i}_m\right) &= \lambda^i\left(s^i\right)\left(S^{-i} \times \left\{\pi^{-i}\right\} \times T^{-i}\right) \\ &= 1 - \varepsilon, \end{aligned}$$

where the first line follows from the induction hypothesis and the second line follows from the definition of $\lambda^i\left(s^i\right)$. ■

Note the following: In the type structure we build, for each i and each $s^i, r^i \in T^i$, if $s^i \neq r^i$ then $\delta^i\left(s^i\right) \neq \delta^i\left(r^i\right)$. To see this, suppose $s^i, r^i \in T^i$ with $s^i \neq r^i$. Fix s^{-i} with $\mu\left(s^i\right)\left(s^{-i}\right) > 0$. Then

$$\delta^i_1\left(s^i\right)\left(s^{-i}, \Gamma\left(s^i; s^{-i}\right)\right) = \varepsilon \times \mu\left(s^i\right)\left(s^{-i}\right) > 0,$$

But $\Gamma\left(r^i; s^{-i}\right) \neq \Gamma\left(s^i; s^{-i}\right)$, so that

$$\delta^i_1\left(r^i\right)\left(s^{-i}, \Gamma\left(s^i; s^{-i}\right)\right) = 0,$$

and so certainly $\delta^i\left(r^i\right) \neq \delta^i\left(s^i\right)$.

Now for the proofs of CI and SUFF. For a given $\left(\tilde{\pi}^j, t^j\right)$, write

$$\left[\tilde{\pi}^j, t^j\right] = S^{-i} \times \left\{\tilde{\pi}^j\right\} \times P^{-i-j} \times \left\{u^j : \delta^j\left(u^j\right) = \delta^j\left(t^j\right)\right\} \times T^{-i-j}.$$

Define $\left[\tilde{\pi}^{-i}, t^{-i}\right]$ similarly. Note, $\left[\tilde{\pi}^j, t^j\right] = (\eta^j_i)^{-1}(\tilde{\pi}^j, (\delta^j \circ \overrightarrow{t}^j_i)(s^{-i}, \tilde{\pi}^{-i}, t^{-i}))$ and $\left[\tilde{\pi}^{-i}, t^{-i}\right] = (\eta^{-i})^{-1}(\tilde{\pi}^{-i}, (\delta^{-i} \circ \overrightarrow{t}_i)(s^{-i}, \tilde{\pi}^{-i}, t^{-i}))$.

Next, fix some $\tilde{\pi}^{-i} = \left(\tilde{\pi}^1, \ldots, \tilde{\pi}^{i-1}, \tilde{\pi}^{i+1}, \ldots, \tilde{\pi}^n\right)$ and $t^{-i} = \left(t^1, \ldots, t^{i-1}, t^{i+1}, \ldots, t^n\right)$ with $\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^{-i}, t^{-i}\right]\right) > 0$. Note, by construction, this implies that $\tilde{\pi}^{-i}$ is equal to π^{-i} or $\Gamma\left(s^i; s^{-i}\right)$.

Proof of CI. By definition

$$\lambda^i\left(s^i\right)\left(\bigcap\nolimits_{k\neq i}\left[s^k\right] \mid \left[\tilde{\pi}^{-i}, t^{-i}\right]\right) = \frac{\lambda^i\left(s^i\right)\left(\bigcap_{k\neq i}\left[s^k\right] \cap \left[\tilde{\pi}^{-i}, t^{-i}\right]\right)}{\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^{-i}, t^{-i}\right]\right)}, \tag{J1}$$

and, for any $j \neq i$,

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \mid \left[\tilde{\pi}^{-i}, t^{-i}\right]\right) = \frac{\lambda^i\left(s^i\right)\left(\left[s^j\right] \cap \left[\tilde{\pi}^{-i}, t^{-i}\right]\right)}{\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^{-i}, t^{-i}\right]\right)}. \tag{J2}$$

First take the case that $t^{-i} = s^{-i}$. Then

$$\begin{aligned}\lambda^i\left(s^i\right)\left(\bigcap_{k\neq i}\left[s^k\right]\cap\left[\tilde{\pi}^{-i},t^{-i}\right]\right) &= \lambda^i\left(s^i\right)\left(\left\{s^{-i}\right\}\times\left\{\tilde{\pi}^{-i}\right\}\times\left\{u^{-i}:\delta^{-i}\left(u^{-i}\right)=\delta^{-i}\left(s^{-i}\right)\right\}\right)\\ &= \lambda^i\left(s^i\right)\left(s^{-i},\tilde{\pi}^{-i},s^{-i}\right) \qquad \text{(J3)}\end{aligned}$$

where the second line uses the fact (shown above) that $\delta^{-i}\left(u^{-i}\right)\neq\delta^{-i}\left(s^{-i}\right)$ if $u^{-i}\neq s^{-i}$. Also,

$$\begin{aligned}\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^{-i},t^{-i}\right]\right) &= \sum_{r^{-i}}\lambda^i\left(s^i\right)\left(\left\{r^{-i}\right\}\times\left\{\tilde{\pi}^{-i}\right\}\times\left\{u^{-i}:\delta^{-i}\left(u^{-i}\right)=\delta^{-i}\left(s^{-i}\right)\right\}\right)\\ &= \sum_{r^{-i}}\lambda^i\left(s^i\right)\left(r^{-i},\tilde{\pi}^{-i},s^{-i}\right)\\ &= \lambda^i\left(s^i\right)\left(s^{-i},\tilde{\pi}^{-i},s^{-i}\right), \qquad \text{(J4)}\end{aligned}$$

where the second equality again uses the fact that $\delta^{-i}\left(u^{-i}\right)\neq\delta^{-i}\left(s^{-i}\right)$ if $u^{-i}\neq s^{-i}$, and the third inequality uses the fact that $\lambda^i\left(s^i\right)\left(r^{-i},\tilde{\pi}^{-i},s^{-i}\right)=0$ when $r^{-i}\neq s^{-i}$. Repeating this argument,

$$\lambda^i\left(s^i\right)\left(\left[s^j\right]\cap\left[\tilde{\pi}^{-i},t^{-i}\right]\right)=\lambda^i\left(s^i\right)\left(s^{-i},\tilde{\pi}^{-i},s^{-i}\right). \qquad \text{(J5)}$$

Putting J1-J5 together yields

$$\lambda^i\left(s^i\right)\left(\bigcap_{k\neq i}\left[s^k\right]\mid\left[\tilde{\pi}^{-i},t^{-i}\right]\right)=1=\prod_{j\neq i}\lambda^i\left(s^i\right)\left(\left[s^j\right]\mid\left[\tilde{\pi}^{-i},t^{-i}\right]\right),$$

so CI holds.

It remains to consider the case $t^{-i}\neq s^{-i}$. Now $\lambda^i\left(s^i\right)\left(s^{-i},\tilde{\pi}^{-i},t^{-i}\right)=0$. Using this and the fact that $u^{-i}\neq t^{-i}$ implies $\delta^{-i}\left(u^{-i}\right)\neq\delta^{-i}\left(t^{-i}\right)$ (shown above), we have that

$$\lambda^i\left(s^i\right)\left(\bigcap_{k\neq i}\left[s^k\right]\cap\left[\tilde{\pi}^{-i},t^{-i}\right]\right)=\lambda^i\left(s^i\right)\left(s^{-i},\tilde{\pi}^{-i},t^{-i}\right)=0. \qquad \text{(J6)}$$

Moreover, $s^j\neq t^j$ for some $j\neq i$, and so we also have

$$\lambda^i\left(s^i\right)\left(\left[s^j\right]\cap\left[\tilde{\pi}^{-i},t^{-i}\right]\right)=0. \qquad \text{(J7)}$$

Putting J1, J6, J2, and J7 together yields

$$\lambda^i\left(s^i\right)\left(\bigcap_{j\neq i}\left[s^j\right]\mid\left[\tilde{\pi}^{-i},t^{-i}\right]\right)=0=\prod_{j\neq i}\lambda^i\left(s^i\right)\left(\left[s^j\right]\mid\left[\tilde{\pi}^{-i},t^{-i}\right]\right),$$

so CI again holds. ■

Proof of SUFF. Fix some $s^j\in S^j$ for $j\neq i$. Also, take $\tilde{\pi}^j$ be the jth coordinate of $\tilde{\pi}^{-i}$. By definition

$$\lambda^i\left(s^i\right)\left(\left[s^j\right]\mid\left[\tilde{\pi}^j,t^j\right]\right)=\frac{\lambda^i\left(s^i\right)\left(\left[s^j\right]\cap\left[\tilde{\pi}^j,t^j\right]\right)}{\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^j,t^j\right]\right)}, \qquad \text{(J8)}$$

and

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \mid \left[\tilde{\pi}^{-i}, t^{-i}\right]\right) = \frac{\lambda^i\left(s^i\right)\left(\left[s^j\right] \cap \left[\tilde{\pi}^{-i}, t^{-i}\right]\right)}{\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^{-i}, t^{-i}\right]\right)}. \tag{J9}$$

First suppose $t^j = s^j$. Recall

$$\lambda^i\left(s^i\right)\left(\left\{r^{-i}\right\} \times \left\{\tilde{\pi}^{-i}\right\} \times \left\{q^{-i} : \delta^{-i}\left(q^{-i}\right) = \delta^{-i}\left(u^{-i}\right)\right\}\right) = 0$$

whenever $r^{-i} \neq u^{-i}$. So, letting $F = \left\{s^j\right\} \times Q^{-i-j}$,

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \cap \left[\tilde{\pi}^j, t^j\right]\right) = \sum\nolimits_{r^{-i} \in F} \lambda^i\left(s^i\right)\left(r^{-i}, \tilde{\pi}^{-i}, r^{-i}\right). \tag{J10}$$

Likewise

$$\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^j, t^j\right]\right) = \sum\nolimits_{r^{-i} \in F} \lambda^i\left(s^i\right)\left(r^{-i}, \tilde{\pi}^{-i}, r^{-i}\right). \tag{J11}$$

Similarly,

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \cap \left[\tilde{\pi}^{-i}, t^{-i}\right]\right) = \lambda^i\left(s^i\right)\left(t^{-i}, \tilde{\pi}^{-i}, t^{-i}\right) \tag{J12}$$

and

$$\lambda^i\left(s^i\right)\left(\left[\tilde{\pi}^{-i}\right] \cap \left[t^{-i}\right]\right) = \lambda^i\left(s^i\right)\left(t^{-i}, \tilde{\pi}^{-i}, t^{-i}\right). \tag{J13}$$

Putting J8-J13 together, we get

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \mid \left[\tilde{\pi}^j, t^j\right]\right) = 1 = \lambda^i\left(s^i\right)\left(\left[s^j\right] \mid \left[\tilde{\pi}^{-i}, t^{-i}\right]\right),$$

so SUFF holds.

Next suppose $t^j \neq s^j$. Then

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \cap \left[\tilde{\pi}^j, t^j\right]\right) = \lambda^i\left(s^i\right)\left(\left[s^j\right] \cap \left[\tilde{\pi}^{-i}, t^{-i}\right]\right) = 0.$$

Together with J8 and J9, it follows

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \mid \left[\tilde{\pi}^j, t^j\right]\right) = 0 = \lambda^i\left(s^i\right)\left(\left[s^j\right] \mid \left[\tilde{\pi}^{-i}, t^{-i}\right]\right),$$

so SUFF again holds. ■

Next, we show that with payoff uncertainty, we lose the analog to the forward direction (part (i)) of Proposition 10.1. For fixed $\varepsilon > 0$, consider the condition of common $(1-\varepsilon)$-belief of an event. We can find a game G and an associated type structure so that the following hold: (i) each type satisfies CI and SUFF; (ii) at any state at which the game is indeed G, there is common $(1-\varepsilon)$-belief of G; (iii) but at any state at which there is RCBR, the strategies played are not correlated rationalizable in G. So, correlated rationalizability does not characterize these conditions. The example below shows how this can happen.

Example J1 *We fix some $\varepsilon > 0$, and consider the two-player game G in Figure J1. Write π^a, π^b for Ann's and Bob's payoff functions in G. We also consider the alternative payoff function $\tilde{\pi}^b$ for Bob given in Figure J2.*

	L	R
U	1, 1	0, 0
D	0, 1	$\frac{1}{\varepsilon}$, 0

Figure J1

	L	R
U	1	0
D	0	1

Figure J2

Construct a type structure as follows. Let $P^a = \{\pi^a\}$, $P^b = \{\pi^b, \tilde{\pi}^b\}$, $T^a = \{t^a\}$, $T^b = \{t^b\}$, and

$$\begin{aligned} \lambda^a(t^a)(L, \pi^b, t^b) &= 1 - \varepsilon, \\ \lambda^a(t^a)(R, \tilde{\pi}^b, t^b) &= \varepsilon, \\ \lambda^b(t^b)(D, \pi^a, t^a) &= 1. \end{aligned}$$

We first show that the game G holds and is common $(1-\varepsilon)$-belief at a state $(\cdot, \pi^a, t^a, \cdot, \pi^b, t^b)$. Letting $E_1^a = S^a \times \{\pi^a\} \times T^a$ and $E_1^b = S^b \times \{\pi^b\} \times T^b$, it suffices to show that $E_m^a = E_1^a$, $E_m^b = E_1^b$, $B^a(E_m^b, \varepsilon) = T^a$, and $B^b(E_m^a, \varepsilon) = T^b$ for all m. For $m = 1$, we have only to note that $\lambda^a(t^a)(E_1^b) = 1 - \varepsilon$ and $\lambda^b(t^b)(E_1^a) = 1$. Assume the statement is true for m. Then

$$E_{m+1}^a = E_m^a \cap [S^a \times P^a \times B^a(E_m^b; \varepsilon)] = E_m^a \cap [S^a \times P^a \times T^a] = E_m^a,$$

and likewise for b, so that

$$\begin{gathered} \lambda^a(t^a)(E_{m+1}^b) = \lambda^a(t^a)(E_m^b) = 1 - \varepsilon, \\ \lambda^b(t^b)(E_{m+1}^a) = \lambda^b(t^b)(E_m^a) = 1, \end{gathered}$$

as required.

Since G is a two-player game, each type certainly satisfies CI and SUFF. Finally, we calculate:

$$\begin{aligned} R_1^a &= \{(D, \pi^a, t^a)\}, \\ R_1^b &= \{(L, \pi^b, t^b), (R, \tilde{\pi}^b, t^b)\}, \end{aligned}$$

from which we get by induction that $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m.

Summing up, each type satisfies CI and SUFF. At the state $(D, \pi^a, t^a, L, \pi^b, t^b)$, the game G holds and is common $(1-\varepsilon)$-belief, and RCBR holds. But the correlated rationalizable set of strategies for G is the singleton $\{(U, L)\}$.

Some comments on the example: First, note that the correlated rationalizable profiles (of the original game G) are even disjoint from the profiles played under our conditions. Certainly, correlated rationalizability does not characterize these conditions.

Also, note that we first fixed ε, and defined common $(1-\varepsilon)$-belief relative to this ε. Then, second, we found a game to show that our conditions need not yield a correlated rationalizable profile. This order is important. Epistemic conditions should be stated independent of a particular game.[2] If the conditions are allowed to depend on the game in question, then the condition could simply be that a strategy profile we are interested in is chosen. This wouldn't be a useful epistemic analysis.

Of course, one reaction is that, given the payoffs in the game of Figure J1, common $(1-\varepsilon)$-belief of this game is a 'significant' amount of uncertainty. But this situation must arise if we fix an epistemic condition to hold over all games, as we have argued we must. The only way to avoid this would be to take ε to be an infinitesimal (and maintain real-valued payoffs). This might be an interesting avenue. To explore it would require reformulating all the apparatus of the paper–type structures etc.–in terms of nonstandard probabilities.

Appendix K Dummy Players

Here we add a dummy player, i.e., a player whose choice of strategy does not affect his payoffs or the payoffs of the other players. In the game with a dummy player, correlated rationalizability gives the same strategies (for non-dummy players) as does correlated rationalizability in the original game. We also show that in any game with a dummy player, correlated rationalizability characterizes RCBR when the additional variables are the hierarchies of beliefs (about the strategies played). Note, a dummy player differs from a move by Nature, in that a dummy player has types.

The definitions are as in the main text. Begin with a given game G. We first extend the game to include a dummy player:

Definition K1 *The game $\overline{G}$ is a dummy extension of $G = \langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \rangle$ if $\overline{G}$ is an $(n+1)$-player strategic-form game $\overline{G} = \langle S^1, \ldots, S^n, S^{n+1}; \overline{\pi}^1, \ldots, \overline{\pi}^n, \overline{\pi}^{n+1} \rangle$ where:*

1. *for $i = 1, \ldots, n$ and all $(s^1, \ldots, s^n, s^{n+1}) \in \prod_{j=1}^{n+1} S^j$,*

$$\overline{\pi}^i \left(s^1, \ldots, s^n, s^{n+1}\right) = \pi^i \left(s^1, \ldots, s^n\right);$$

2. *for all $(s^1, \ldots, s^n, s^{n+1})$, $(r^1, \ldots, r^n, r^{n+1}) \in \prod_{j=1}^{n+1} S^j$,*

$$\overline{\pi}^{n+1} \left(s^1, \ldots, s^n, s^{n+1}\right) = \overline{\pi}^{n+1} \left(r^1, \ldots, r^n, r^{n+1}\right).$$

[2] The same order of quantification (in a non-epistemic setting) is chosen in Weibull [3, 1992].

(This says that the game is decomposable into the player set $\{1, \ldots, n\}$ and the player $n+1$. Cf. Mertens [1, 1989, p.577].) When given a game G and a dummy extension $\overline{G}$, we will take $S = \prod_{i=1}^{n} S^i$, $S^{-i} = \prod_{j \neq i: j=1,\ldots,n} S^j$, etc. (That is, define the sets as they would be in G.) Begin by relating the correlated rationalizable strategies in G to the correlated rationalizable strategies in an extension $\overline{G}$.

Lemma K1 *Fix a game G and a dummy extension $\overline{G}$ of G. If $S_M^1 \times \cdots \times S_M^n \times S_M^{n+1}$ are the correlated rationalizable strategies in $\overline{G}$, then $S_M^1 \times \cdots \times S_M^n$ are the correlated rationalizable strategies in G.*

Proof. Write $Q = Q^1 \times \cdots \times Q^n$ for the correlated rationalizable strategies in G. We will show that $Q = S_M$. Note, in any extended game, $S_M^{n+1} = S^{n+1}$. We will make use of this fact below.

We first show that $Q \times S^{n+1}$ is a BRS in $\overline{G}$, from which it follows that $Q \times S^{n+1} \subseteq S_M \times S^{n+1}$. Fix $i = 1, \ldots, n$ and some $s^i \in Q^i$. Since the correlated rationalizable set is a BRS, there is a $\mu \in \mathcal{M}\left(S^{-i}\right)$ with $\mu\left(Q^{-i}\right) = 1$ and $\pi^i\left(s^i, \mu\right) \geq \pi^i\left(r^i, \mu\right)$ for all $r^i \in S^i$. Pick some $\nu \in \mathcal{M}\left(S^{-i} \times S^{n+1}\right)$ so that $\text{marg}_{S^{-i}} \nu = \mu$. By definition, $\nu\left(Q^{-i} \times S^{n+1}\right) = 1$. Also, for any $r^i \in S^i$, $\overline{\pi}^i\left(r^i, \nu\right) = \pi^i\left(r^i, \mu\right)$. With this, $\overline{\pi}^i\left(s^i, \nu\right) \geq \overline{\pi}^i\left(r^i, \nu\right)$ for all $r^i \in S^i$. Next fix $s^{n+1} \in S^{n+1}$ and note that it is optimal under any measure on Q. This establishes that $Q \times S^{n+1}$ is a BRS in $\overline{G}$.

Next, we will show that S_M is a BRS in G, from which it follows that $S_M \subseteq Q$. Fix $i = 1, \ldots, n$ and some $s^i \in S_M^i$. Then there is a $\nu \in \mathcal{M}\left(S^{-i} \times S^{n+1}\right)$ with $\nu\left(S_M^{-i} \times S^{n+1}\right) = 1$ and $\overline{\pi}^i\left(s^i, \nu\right) \geq \overline{\pi}^i\left(r^i, \nu\right)$ for all $r^i \in S^i$. Let $\mu = \text{marg}_{S^{-i}} \nu$. Then $\mu\left(S_M^{-i}\right) = 1$. Moreover, for any $r^i \in S^i$, $\overline{\pi}^i\left(r^i, \nu\right) = \pi^i\left(r^i, \mu\right)$. From this, $\pi^i\left(s^i, \mu\right) \geq \pi^i\left(r^i, \mu\right)$ for all $r^i \in S^i$, establishing the result. ■

Now we show that in a game with dummy players, the correlated rationalizable strategies characterize CI, SUFF, and RCBR. (Here, CI and SUFF are Definitions 9.1-9.2 in the main text.)

Proposition K1 *Fix a game G and a dummy extension $\overline{G}$ of G.*

(i) Fix a type structure for $\overline{G}$ and a state $\left(s^1, t^1, \ldots, s^{n+1}, t^{n+1}\right)$. If each type t^i satisfies CI and SUFF, and RCBR holds, then the profile $\left(s^1, \ldots, s^{n+1}\right)$ is correlated rationalizable in $\overline{G}$.

(ii) There is a type structure for $\overline{G}$ such that, for each correlated rationalizable strategy profile $\left(s^1, \ldots, s^{n+1}\right)$ in $\overline{G}$, there is a state $\left(s^1, t^1, \ldots, s^{n+1}, t^{n+1}\right)$ at which at which each type t^i satisfies CI and SUFF, and RCBR holds.

Part (i) is immediate from Proposition 10.1.

For part (ii): First note that if $|S_M| = 1$, the result is trivial. (Follow the proof of Proposition 10.1(ii) in the main text and then observe that the type structure trivially satisfies CI and SUFF.) So, assume $|S_M| \geq 2$.

Let $S_M^1 \times \cdots \times S_M^{n+1}$ be the correlated rationalizable strategies in $\overline{G}$, and note that $S_M^{n+1} = S^{n+1}$. For $i = 1, \ldots, n$, take $T^i = S_M^i$. Take T^{n+1} so that $\left|T^{n+1}\right| = \max\left\{\left|S_M^i\right| : i = 1, \ldots, n\right\} \times \left|S^{n+1}\right|$. Then, for each $i = 1, \ldots, n$, we can find an injective map $g^i : S_M^i \times S^{n+1} \to T^{n+1}$.

For each $i = 1, \ldots, n$ and each $s^i \in T^i = S^i_M$, define the map $s^i \to \mu\left(s^i\right) \in \mathcal{M}\left(S^{-i} \times S^{n+1}\right)$ so that $\mu\left(s^i\right)\left(S^{-i}_M \times S^{n+1}_M\right) = 1$ and s^i is optimal under $\mu\left(s^i\right)$. Construct λ^i so that, for each $s^i \in T^i = S^i_M$,

$$\lambda^i\left(s^i\right)\left(s^{-i}, s^{n+1}, t^{-i}, t^{n+1}\right) = \begin{cases} \mu\left(s^i\right)\left(s^{-i}, s^{n+1}\right) & \text{if } s^{-i} = t^{-i} \text{ and } g^i\left(s^i, s^{n+1}\right) = t^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

For each $t^{n+1} \in T^{n+1}$, define an injective map $t^{n+1} \mapsto \mu\left(t^{n+1}\right) \in \mathcal{M}\left(S^{n+1}\right)$ with $\mu\left(t^{n+1}\right)(S_M) = 1$. This can be done since $|S_M| \geq 2$. Then define λ^{n+1} so that

$$\lambda^{n+1}\left(t^{n+1}\right)(s, t) = \begin{cases} \mu\left(t^{n+1}\right)(s) & \text{if } s = t, \\ 0 & \text{otherwise.} \end{cases}$$

We now turn to showing the properties.

Proof of RCBR. We will show that $S^1_M \times \cdots \times S^{n+1}_M \subseteq \text{proj}_S\, R_m$, for all m. For this, it suffices to show that, for all m: (im) for $i = 1, \ldots, n$, $\left\{\left(s^i, s^i\right) \in S^i \times T^i : s^i \in S^i_M\right\} \subseteq R^i_m$; and (ii$m$) $S^{n+1} \times T^{n+1} = R^{n+1}_m$.

Begin with $m = 1$. For $i = 1, \ldots, n$, if $s^i \in S^i_M$ then $\left(s^i, s^i\right) \in R^i_1$ so that (i1) holds. Part (ii1) is immediate from the construction of an extended game. Now assume (im) and (ii$m$) hold. Fix $i = 1, \ldots, n$ and $s^i \in S^i_M$. Then

$$\lambda^i\left(s^i\right)\left(\left\{\left(s^{-i}, s^{-i}\right) : s^{-i} \in S^{-i}_M\right\} \times S^{n+1} \times T^{n+1}\right) = 1,$$

establishing that $\left(s^i, s^i\right) \in R^i_{m+1}$ and (i$(m+1)$). For (ii$(m+1)$), we only need show that, for any $t^{n+1} \in T^{n+1}$, $t^{n+1} \in B^{n+1}(R_m)$. But

$$\lambda^{n+1}\left(s^{n+1}\right)\left(\left(s^1, s^1, \ldots, s^n, s^n\right) : s^i \in S^i_M\right\}\right) = 1,$$

from which this is immediate. ■

Note the following: For each i, if $t^i \neq u^i$ then $\delta^i\left(t^i\right) \neq \delta^i\left(u^i\right)$. To show this, begin with player $n+1$ and fix types $t^{n+1}, u^{n+1} \in T^{n+1}$. By construction, $\text{marg}_S\, \lambda^{n+1}\left(t^{n+1}\right) \neq \text{marg}_S\, \lambda^{n+1}\left(u^{n+1}\right)$, so that $\delta^{n+1}_1\left(t^{n+1}\right) \neq \delta^{n+1}_1\left(u^{n+1}\right)$, as required.

Now turn to some player $i = 1, \ldots, n$ and fix $t^i, u^i \in T^i$ with $t^i \neq u^i$. Also, fix a profile $\left(s^{-i}, s^{n+1}, t^{-i}, t^{n+1}\right) \in \text{Supp}\, \lambda^i\left(t^i\right)$. Note, $g^i\left(t^i, s^{n+1}\right) = t^{n+1}$. So, for any $r^{n+1} \in S^{n+1} \backslash \left\{s^{n+1}\right\}$, $g^i\left(u^i, r^{n+1}\right) \neq t^{n+1}$. By construction,

$$\lambda^i\left(u^i\right)\left(S^{-i} \times S^{n+1} \times T^{-i} \times \left\{u^{n+1} : \delta^{n+1}_1\left(u^{n+1}\right) = \delta^{n+1}_1\left(t^{n+1}\right)\right\}\right) = 0.$$

Next, fix $E = S^{-i} \times S^{n+1} \times \prod_{j \neq i; j=1,..,n} \mathcal{M}\left(S^{-j}\right) \times \left\{\delta_1^{n+1}\left(t^{n+1}\right)\right\}$. Then

$$\begin{aligned}
\underline{\rho}_2^i\left(\lambda^i\left(t^i\right)\right)(E) &= \lambda^i\left(t^i\right)\left(\left(\rho_2^i\right)^{-1}(E)\right) \\
&= \lambda^i\left(t^i\right)\left(S^{-i} \times S^{n+1} \times T^{-i} \times \left\{v^{n+1} : \delta_1^{n+1}\left(v^{n+1}\right) = \delta_1^{n+1}\left(t^{n+1}\right)\right\}\right) \\
&> 0 \\
&= \lambda^i\left(u^i\right)\left(S^{-i} \times S^{n+1} \times T^{-i} \times \left\{v^{n+1} : \delta_1^{n+1}\left(v^{n+1}\right) = \delta_1^{n+1}\left(t^{n+1}\right)\right\}\right) \\
&= \lambda^i\left(u^i\right)\left(\left(\rho_2^i\right)^{-1}(E)\right) = \underline{\rho}_2^i\left(\lambda^i\left(u^i\right)\right)(E).
\end{aligned}$$

So, $\underline{\rho}_2^i\left(\lambda^i\left(t^i\right)\right) \neq \underline{\rho}_2^i\left(\lambda^i\left(u^i\right)\right)$, i.e. $\delta_2^i\left(t^i\right) \neq \delta_2^i\left(u^i\right)$.

Proof of CI. Fix a player i and some $s^i \in T^i = S_M^i$. Let $\left(s^{-i}, s^{n+1}\right) = \left(s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^{n+1}\right)$ and $\left(t^{-i}, t^{n+1}\right) = \left(t^1, \ldots, t^{i-1}, t^{i+1}, \ldots, t^{n+1}\right)$. Suppose $\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right) > 0$. If either $t^j \neq s^j$, for some $j = 1, \ldots, n$ with $j \neq i$, or $g^i\left(s^i, s^{n+1}\right) \neq t^{n+1}$, then s^i trivially satisfies CI and SUFF. So suppose that, for all $j = 1, \ldots, n$ with $j \neq i$, $t^j = s^j$ and $g^i\left(s^i, s^{n+1}\right) = t^{n+1}$. Then

$$\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[s^j\right] \cap \left[s^{n+1}\right] \mid \bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right) = \frac{\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right)}{\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right)} = 1.$$

Also, for each $j \neq i$, $j = 1, \ldots, n, n+1$,

$$\lambda^i\left(s^i\right)\left(\left[s^j\right] \mid \bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right) = \frac{\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right)}{\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right)} = 1.$$

From this it follows that s^i satisfies CI.

To show type $t^{n+1} \in T^{n+1}$ satisfies CI note that, for each $i = 1, \ldots, n$, $\delta^i\left(t^i\right) \neq \delta^i\left(u^i\right)$ whenever $t^i \neq u^i$. (This was shown above.) So, CI follows from Proposition H1. ■

Proof of SUFF. Fix i and some $s^i \in T^i = S_M^i$. Let $\left(s^{-i}, s^{n+1}\right) = \left(s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^{n+1}\right)$ and $\left(t^{-i}, t^{n+1}\right) = \left(t^1, \ldots, t^{i-1}, t^{i+1}, \ldots, t^{n+1}\right)$. Suppose $\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right) > 0$. If either $t^j \neq s^j$, for some $j = 1, \ldots, n$ with $j \neq i$, or $g^i\left(s^i, s^{n+1}\right) = t^{n+1}$, then s^i trivially satisfies CI and SUFF. So suppose that, for all $j = 1, \ldots, n$ with $j \neq i$, $t^j = s^j$ and $g^i\left(s^i, s^{n+1}\right) = t^{n+1}$.

Fix $k = 1, \ldots, n$ and note that

$$\lambda^i\left(s^i\right)\left(\left[s^k\right] \mid \left[t^k\right]\right) = \frac{\lambda^i\left(s^i\right)\left(\left[t^k\right]\right)}{\lambda^i\left(s^i\right)\left(\left[t^k\right]\right)} = 1.$$

Moreover,

$$\lambda^i\left(s^i\right)\left(\left[s^k\right] \mid \bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right) = \frac{\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right)}{\lambda^i\left(s^i\right)\left(\bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right)} = 1.$$

So $\lambda^i\left(s^i\right)\left(\left[s^k\right] \mid \bigcap_{j \neq i}\left[t^j\right] \cap \left[t^{n+1}\right]\right) = \lambda^i\left(s^i\right)\left(\left[s^k\right] \mid \left[t^k\right]\right)$ and s^i satisfies SUFF.

To show type $t^{n+1} \in T^{n+1}$ satisfies SUFF, again note that, for each $i = 1, \ldots, n$, $\delta^i\left(t^i\right) \neq \delta^i\left(u^i\right)$ whenever $t^i \neq u^i$. So, SUFF follows from Proposition H1. ■

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