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ADMISSIBILITY IN GAMES

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ADMISSIBILITY IN GAMES¹

BY ADAM BRANDENBURGER, AMANDA FRIEDENBERG,
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Suppose that each player in a game is rational, each player thinks the other players are rational, and so on. Also, suppose that rationality is taken to incorporate an admissibility requirement—that is, the avoidance of weakly dominated strategies. Which strategies can be played? We provide an epistemic framework in which to address this question. Specifically, we formulate conditions of rationality and m th-order assumption of rationality ($RmAR$) and rationality and common assumption of rationality (RCAR). We show that (i) RCAR is characterized by a solution concept we call a “self-admissible set”; (ii) in a “complete” type structure, $RmAR$ is characterized by the set of strategies that survive $m + 1$ rounds of elimination of inadmissible strategies; (iii) under certain conditions, RCAR is impossible in a complete structure.

KEYWORDS: Epistemic game theory, rationality, admissibility, iterated weak dominance, self-admissible sets, assumption, completeness.

1. INTRODUCTION

WHAT IS THE IMPLICATION of supposing that each player in a game is rational, each player thinks the other players are rational, and so on? The natural first answer to this question is that the players will choose *iteratively undominated* (IU) strategies—that is, strategies that survive iterated deletion of strongly dominated strategies. Bernheim (1984) and Pearce (1984) gave essentially this answer, via their concept of rationalizability.² Pearce (1984) also defined the concept of a *best-response set* (BRS) and gave this as a more complete answer.

In this paper we ask: What is the answer to the above question when rationality of a player is taken to incorporate an admissibility requirement—that is, the avoidance of weakly dominated strategies?

¹This paper combines two earlier papers, “Epistemic Conditions for Iterated Admissibility” (by Brandenburger and Keisler, June 2000) and “Common Assumption of Rationality in Games” (by Brandenburger and Friedenberg, January 2002). We are indebted to Bob Aumann, Pierpaolo Battigalli, Martin Cripps, Joe Halpern, Johannes Hörner, Martin Osborne, Marciano Siniscalchi, and Gus Stuart for important input. Geir Asheim, Chris Avery, Oliver Board, Giacomo Bonanno, Ken Corts, Lisa DeLucia, Christian Ewerhart, Konrad Grabiszewski, Rena Henderson, Elon Kohlberg, Stephen Morris, Ben Polak, Phil Reny, Dov Samet, Michael Schwarz, Jeroen Swinkels, and participants in various seminars gave valuable comments. Eddie Dekel and the referees made very helpful observations and suggestions. Brandenburger gratefully acknowledges support from Harvard Business School and the Stern School of Business. Friedenberg thanks the CMS-EMS at Northwestern University, the Department of Economics at Yale University, and the Olin School of Business. Keisler thanks the National Science Foundation and the Vilas Trust Fund.

²Under the original definition, which makes an independence assumption, the rationalizable strategies can be a strict subset of the iteratively undominated strategies. Recent definitions (e.g., Osborne and Rubinstein (1994)) allow for correlation; in this case, the two sets are equal.

Our analysis will identify a weak-dominance analog to Pearce's concept of a BRS, which we call a self-admissible set (SAS). We will also identify conditions under which players will choose iteratively admissible (IA) strategies—that is, strategies that survive iterated deletion of weakly dominated strategies.

The case of weak dominance is important. Weak-dominance concepts give sharp predictions in many games of applied interest. Separate from its power in applications, admissibility is a *prima facie* reasonable criterion: It captures the idea that a player takes all strategies for the other players into consideration; none is entirely ruled out. It also has a long heritage in decision and game theory. (See the discussion in Kohlberg and Mertens (1986, Section 2.7).)

But there are significant conceptual hurdles to overcome in order understand admissibility in games. Below, we review some issues that have already been identified in the literature, add new ones, and offer a resolution.

The paper is organized as follows. The next section is an informal discussion of the issues and results to follow. The formal treatment is in Sections 3–10. Section 11 discusses some open questions. The heuristic treatment of the next section can be read either before or in parallel with the formal treatment.³

2. HEURISTIC TREATMENT

We begin with the standard equivalence: Strategy s is admissible if and only if there is a strictly positive probability measure on the strategy profiles for the other players, under which s is optimal. In an influential paper, Samuelson (1992) pointed out that this poses a basic challenge for an analysis of admissibility in games. Consider the game in Figure 2.1, which is essentially Example 8 in Samuelson (1992).

Suppose rationality incorporates admissibility. Then, if Bob is rational, he should assign positive probability to both U and D , and so will play L . Likewise, if Ann is rational, presumably she should assign positive probability to both L and R . But if Ann thinks Bob is rational, should she not assign probability 1 to L ? (We deliberately use the loose term “thinks.” We will be more

		Bob	
		L	R
	U	1, 1	0, 1
Ann	D	0, 2	1, 0

FIGURE 2.1.

³Online Supplemental material can be found in Brandenburger, Friedenber, and Keisler (2008).

precise below.) The condition that Ann is rational appears to conflict with the condition that she thinks Bob is rational.

2.1. Lexicographic Probabilities

Our method for overcoming this hurdle will be to allow Ann at the same time both to include and to exclude a strategy of Bob's. Ann will consider some of Bob's strategies infinitely less likely than others, but still possible. The strategies that get infinitesimal weight can be viewed as both included (because they do not get zero weight) and excluded (because they get only infinitesimal weight).

In Figure 2.2, Ann has a *lexicographic probability system* (LPS) on Bob's strategies. (LPS's were introduced in Blume, Brandenburger, and Dekel (1991a).) Ann's primary measure ("hypothesis") assigns probability 1 to L . Her secondary measure (depicted in square brackets) assigns probability 1 to R . Ann considers it infinitely more likely that Bob plays L than that Bob plays R , but does not entirely exclude R from consideration.

In our lexicographic decision theory, Ann will choose strategy s over strategy s' if s yields a sequence of expected payoffs lexicographically greater than the sequence s' yields. So, with the LPS shown, she will choose U (not D).

Can we say that Ann believes Bob is rational? Customarily, we would say yes if Ann assigns probability 1 to the event that Bob is rational, but now Ann has an LPS that is not a single measure, so we need to look at the question at a deeper level. Recall, at the level of preferences, Ann believes an event E if her preference over acts, conditional on not- E , is trivial. (In short, not- E is Savage-null.) But, clearly, Ann's preference conditional on the event that Bob is irrational (plays R) is not trivial: under her secondary hypothesis, she chooses D over U .

We will settle for the weaker condition that Ann considers the event E infinitely more likely than not- E and, in this case, we will say Ann *assumes* E . (Later, we give assumption a preference basis.) In Figure 2.2, Ann considers it infinitely more likely that Bob is rational than irrational. This is our resolution of the tension between requiring Ann to be rational—in the sense of admissibility—and requiring her to think Bob is rational.

	1	[1]
	L	R
U	1, 1	0, 1
D	0, 2	1, 0

FIGURE 2.2.

LPS's are a basic tool for dealing with the idea of unexpected events in the context of a strategic-form analysis. There is an analogous tool for analyzing the extensive form, namely conditional probability systems (CPS's). (The concept goes back to Rényi (1955).) CPS's are a key element of the Battigalli and Siniscalchi (2002) (henceforth B-S) extensive-form epistemic analysis. Our (LPS-based) assumption concept is closely related to their (CPS-based) concept of "strong belief." In fact, there will be a close parallel between many of the ingredients in our paper and in B-S. Section 2.8 returns to discuss these connections and to highlight the big debt we owe to B-S.

2.2. Rationality and Common Assumption of Rationality

In the game in Figure 2.2, the conditions that Bob is rational, and that Ann is rational and assumes Bob is rational, imply a unique strategy for each player. In general, we can formulate an infinite sequence of conditions:

- | | |
|--|--|
| (a1) Ann is rational; | (b1) Bob is rational; |
| (a2) Ann is rational and assumes (b1); | (b2) Bob is rational and assumes (a1); |
| (a3) Ann is rational, assumes (b1),
assumes (b2); | (b3) Bob is rational, assumes (a1),
assumes (a2); |
| ... | ... |

There is *rationality and common assumption of rationality* (RCAR) if this sequence holds. RCAR is a natural "baseline" epistemic condition on a game when rationality incorporates admissibility. We want to know what strategies can be played under RCAR.

To answer, we need some more epistemic apparatus. Let T^a and T^b be spaces of *types* for Ann and Bob, respectively. Each type t^a for Ann is associated with an LPS on the product of Bob's strategy and type spaces (i.e., on $S^b \times T^b$). Likewise for Bob. A state of the world is a 4-tuple (s^a, t^a, s^b, t^b) , where s^a and t^a are Ann's actual strategy and type, and likewise for Bob. This is a standard *type structure* in the epistemic literature, with the difference that types are associated with LPS's, not single probability measures.

In these structures, rationality is a property of a strategy–type pair. A pair (s^a, t^a) is *rational* if it satisfies the following *admissibility* requirement: The LPS σ associated with t^a has full support (rules nothing out), and s^a lexicographically maximizes Ann's expected payoff under σ . (In particular, s^a is not weakly dominated.) Otherwise the pair is irrational. Likewise for Bob.

Start with a game and an associated type structure. We get a picture like Figure 2.3, where the outer rectangle is $S^b \times T^b$ and the shaded area is the strategy–type pairs satisfying "RCAR for Bob."

Now fix a strategy–type pair (s^a, t^a) that satisfies RCAR for Ann. Then Ann assumes (b1), assumes (b2), ... By a conjunction property of assumption, it follows that Ann assumes the joint event (b1) and (b2) and ..., that is, Ann assumes "RCAR for Bob." This gives a picture like Figure 2.4, where the se-

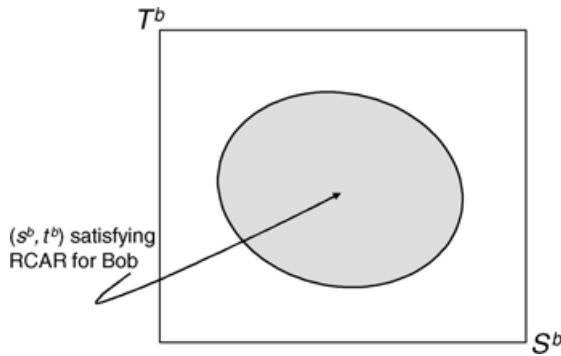


FIGURE 2.3.

quence of measures $(\mu_0, \dots, \mu_{n-1})$ is the LPS associated with t^a . There is an initial segment (μ_0, \dots, μ_j) of this sequence which concentrates exactly on the event “RCAR for Bob.” This is because Ann considers pairs (s^b, t^b) inside this event infinitely more likely than pairs outside the event.

Since (s^a, t^a) is rational, strategy s^a lexicographically maximizes Ann’s expected payoff, under $(\mu_0, \dots, \mu_{n-1})$. This establishes (by taking a convex combination of the marginals on S^b) that there is a strictly positive measure on S^b under which s^a is optimal. That is, s^a must be admissible. Strategy s^a must also lexicographically maximize Ann’s expected payoff under the initial segment (μ_0, \dots, μ_j) . It follows (again taking a convex combination of the marginals) that there is a strictly positive measure on the projection of the event “RCAR for Bob” under which s^a is optimal. That is, s^a must be admissible with respect to the projection.

Take the set of all states (s^a, t^a, s^b, t^b) satisfying RCAR and let $Q^a \times Q^b$ be its projection into $S^a \times S^b$. By the discussion above, the product $Q^a \times Q^b$ has the following two properties:

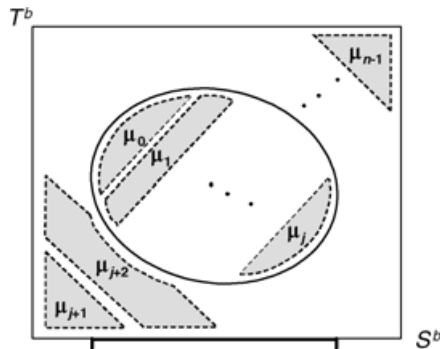


FIGURE 2.4.

- (i) each $s^a \in Q^a$ is admissible (i.e., is admissible with respect to S^b),
 (ii) each $s^a \in Q^a$ is admissible with respect to Q^b ;
 and likewise with a and b interchanged.

(Note the similarity of these properties to the definition of a best-response set (Pearce (1984))—a concept based, of course, on strong dominance.) But these two properties are not yet enough to characterize RCAR, as the next example shows.

2.3. Convex Combinations

Consider the game in Figure 2.5. The set $\{U\} \times \{L, R\}$ has properties (i) and (ii), but U cannot be played under RCAR. Indeed, fix a type structure and suppose (U, t^a) is rational. In terms of Ann's payoffs, $U = \frac{1}{2}N + \frac{1}{2}D$. It follows that (N, t^a) and (D, t^a) will also be rational. Next consider a strategy-type pair (s^b, t^b) for Bob, which is rational and assumes Ann is rational (i.e., Bob assumes the event (a1) defined in Section 2.2). So Bob considers rational pairs (s^a, t^a) for Ann infinitely more likely than irrational pairs. But then $s^b = R$, since the LPS associated with t^b must give each of U, N , and D positive probability before giving M positive probability. Now consider a strategy-type pair (s^a, t^a) for Ann, which is rational and such that Ann assumes Bob is rational and assumes Bob is rational and assumes Ann is rational (that is, Ann assumes the events (b1) and (b2)). We get $s^a = D$ (not U), since the LPS associated with t^a must give R positive probability before giving L positive probability.

The key to the example is that U is a convex combination for Ann of N and D , so that (N, t^a) and (D, t^a) are rational whenever (U, t^a) is. This suggests that the projection of the RCAR set should have the following property:

- (iii) if $s^a \in Q^a$ and r^a is part of a convex combination of strategies for Ann that is equivalent for her to s^a , then $r^a \in Q^a$;
 and likewise for Bob.

		Bob	
		L	R
Ann	U	1, 4	1, 4
	M	-1, 3	-1, 0
	N	2, 0	0, 3
	D	0, 0	2, 3

FIGURE 2.5.

We define a *self-admissible set* (SAS) to be a set $Q^a \times Q^b \subseteq S^a \times S^b$ of strategy pairs which has properties (i), (ii), and (iii).⁴ The strategies played under RCAR always constitute an SAS (Theorem 8.1(i)).

2.4. Irrationality

Does the converse hold? That is, given an SAS, is there an associated type structure so that the strategies played under RCAR correspond to this SAS?

To address the converse direction, we need to consider a further aspect of admissibility in games. Under admissibility, Ann considers everything possible. But this is only a decision-theoretic statement. Ann is in a game, so we imagine she asks herself: “What about Bob? What does he consider possible?” If Ann truly considers everything possible, then it seems she should, in particular, allow for the possibility that Bob does not! Alternatively put, it seems that a full analysis of the admissibility requirement should include the idea that other players do not conform to the requirement.

More precisely, we know that if a strategy–type pair (s^a, t^a) for Ann is rational, then the LPS associated with t^a has full support. But we are going to allow Ann to consider the possibility that there are types t^b for Bob associated with LPS’s that do not have full support. (Ann allows that Bob does not consider everything.) Of course, by definition, if (s^b, t^b) is a rational pair for Bob, then the LPS associated with t^b will have full support. But there may be other strategy–type pairs present too. Our argument is that the presence of such pairs is conceptually appropriate if the topic is admissibility in games.

To see the significance of this, consider the game in Figure 2.6 (kindly provided by Pierpaolo Battigalli). The set $\{U, M, D\} \times \{C, R\}$ is an SAS. (It is also the IA set.) With the converse direction in mind, let us understand why D is consistent with RCAR.

		Bob		
		L	C	R
Ann	U	4, 0	4, 1	0, 1
	M	0, 0	0, 1	4, 1
	D	3, 0	2, 1	2, 1

FIGURE 2.6.

⁴Brandenburger and Friedenberg (2004) investigated properties of SAS’s.

Fix a type structure. Notice that L is (strongly) dominated, so all pairs (L, t^b) for Bob are irrational. A pair (C, t^b) or (R, t^b) will be rational if the LPS associated with t^b has full support, and irrational otherwise. We use this below.

Turn to Ann. Notice that if D is optimal under a measure, then the measure either assigns probability $\frac{1}{2}$ to C and $\frac{1}{2}$ to R or assigns positive probability to both L and R . Moreover, in the first case, U and M will necessarily be optimal too.

Fix a rational pair (D, t^a) , where t^a assumes Bob is rational. Let $(\mu_0, \dots, \mu_{n-1})$ be the full-support LPS associated with t^a . By the full-support condition, there is some measure that gives $\{L\} \times T^b$ positive probability. Let μ_i be the first such measure. Also, since t^a assumes “Bob is rational,” the rational strategy–type pairs for Bob must be infinitely more likely than the irrational pairs. Therefore $i \neq 0$. Using the rationality of (D, t^a) , we now have that for each measure μ_k with $k < i$: (i) μ_k assigns probability $\frac{1}{2}$ to $\{C\} \times T^b$ and probability $\frac{1}{2}$ to $\{R\} \times T^b$, and (ii) U, M , and D are each optimal under μ_k . It follows that D must also be optimal under μ_i , and so μ_i must assign positive probability to both $\{L\} \times T^b$ and $\{R\} \times T^b$. Now use again the fact that rational strategy–type pairs for Bob must be infinitely more likely than the irrational pairs. Since each point in $\{L\} \times T^b$ is irrational, μ_i must assign strictly positive probability to the irrational pairs in $\{R\} \times T^b$. This is possible if there are non-full-support types for Bob.

It is important to understand that we have two forms of irrationality in this paper. One is more or less standard: A strategy–type pair is irrational if s^a is not optimal under the LPS associated with t^a . This is just the usual notion of irrationality, but now optimality is defined lexicographically. In a type structure, some strategy–type pairs “do their sums right” and optimize, and others do not. Both kinds of pairs are present, but the latter kind do not play a special role in our analysis.

There is a second form of irrationality which is new. For us, a player is rational if he optimizes and also rules nothing out. So irrationality might mean not optimizing. But it can also mean optimizing while not considering everything possible (the LPS associated with t^a does not have full support). This form of irrationality is present in the example above, and it plays a central role in our analysis. (See also Section 11C.)

To keep things simple, we use the one term “irrationality” to cover both situations, but we repeat that there are these two cases.

2.5. Characterization of RCAR

We can now state our characterization of RCAR in games (Theorem 8.1):

Start with a game and an associated type structure. Let $Q^a \times Q^b$ be the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying RCAR. Then $Q^a \times Q^b$ is an SAS of the game.

		Bob	
		L	R
Ann	U	2, 2	2, 2
	M	1, 1	0, 0
	D	1, 1	3, 3

FIGURE 2.7.

We also have:

Start with a game and an SAS $Q^a \times Q^b$. There is a type structure (with non-full-support types) such that $Q^a \times Q^b$ is the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying RCAR.

It is easy to check that the IA strategies constitute an SAS of a game. So, in particular, every game possesses an SAS, and RCAR is possible in every game. But a game may possess other SAS's too. In the game in Figure 2.7, there are three SAS's: $\{(U, L)\}$, $\{U\} \times \{L, R\}$, and $\{(D, R)\}$. (The third is the IA set. Note that the other two SAS's are not contained in the IA set. This is different from the case of strong dominance: It is well known that any Pearce best-response set is contained in the set of strategies that survives iterated strong dominance.)

2.6. Iterated Admissibility

So the question remains: What epistemic conditions select the IA set in a game from among the family of SAS's? To investigate this, consider Figure 2.8, which gives a type structure for the game in Figure 2.7. Ann and Bob each have a single type. Ann's LPS assigns primary probability 1 to (L, t^b) , and secondary probability 1 (in square brackets) to (R, t^b) . Bob's LPS assigns primary probability 1 to (U, t^a) , secondary probability 1 (in square brackets) to (M, t^a) , and tertiary probability 1 (in double square brackets) to (D, t^a) . Ann (resp. Bob) has just one rational strategy–type pair, namely (U, t^a) (resp. (L, t^b)). Ann's unique type t^a assumes Bob is rational (the rational pair (L, t^b) is considered infinitely more likely than the irrational pair (R, t^b)). Likewise, Bob's

<p>LPS associated with t^a:</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td style="padding: 5px;">T^b t^b</td> <td style="padding: 5px; text-align: center;">1</td> <td style="padding: 5px; text-align: center;">[1]</td> </tr> <tr> <td></td> <td style="text-align: center; padding: 5px;">L</td> <td style="text-align: center; padding: 5px;">R</td> </tr> <tr> <td></td> <td colspan="2" style="text-align: center; padding: 5px;">S^b</td> </tr> </table>	T^b t^b	1	[1]		L	R		S^b		<p>LPS associated with t^b:</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td style="padding: 5px;">T^a t^a</td> <td style="padding: 5px; text-align: center;">1</td> <td style="padding: 5px; text-align: center;">[1]</td> <td style="padding: 5px; text-align: center;">[[1]]</td> </tr> <tr> <td></td> <td style="text-align: center; padding: 5px;">U</td> <td style="text-align: center; padding: 5px;">M</td> <td style="text-align: center; padding: 5px;">D</td> </tr> <tr> <td></td> <td colspan="3" style="text-align: center; padding: 5px;">S^a</td> </tr> </table>	T^a t^a	1	[1]	[[1]]		U	M	D		S^a		
T^b t^b	1	[1]																				
	L	R																				
	S^b																					
T^a t^a	1	[1]	[[1]]																			
	U	M	D																			
	S^a																					

FIGURE 2.8.

unique type t^b assumes Ann is rational (the rational pair (U, t^a) is considered infinitely more likely than the irrational pairs (M, t^a) and (D, t^a)). By induction, the RCAR set is then the singleton $\{(U, t^a, L, t^b)\}$. This is an instance of Theorem 8.1: The projection into $S^a \times S^b$ of $\{(U, t^a, L, t^b)\}$ is an SAS, namely $\{(U, L)\}$.

In this structure, Ann assumes Bob plays L , making U her unique rational choice. Both M and D are irrational for her. In fact, Bob considers it infinitely more likely that Ann plays M than D , which is why he plays L . Bob is free to assign the probabilities this way. To assume Ann is rational, it is enough that Bob considers U infinitely more likely than both M and D , as he does.

What if Bob considered D infinitely more likely than M ? Then he would rationally play R not L . Presumably Ann would then play D and the IA set would result. Figure 2.9 gives a scenario under which Bob will, in fact, consider D infinitely more likely than M . We have added a type u^a for Ann that assumes Bob plays R . Now there is a second rational pair for Ann, namely (D, u^a) . (Note there is no type v^a for Ann which we could add to the structure to make (M, v^a) rational for Ann, since M is inadmissible.) If Bob assumes Ann is rational, then he must consider the shaded pairs in Figure 2.9 infinitely more likely than the unshaded pairs. If rational, he must play R , as desired.

Call a type structure *complete* if the range of the map from T^a (Ann’s type space) to the space of LPS’s on $S^b \times T^b$ (Bob’s strategy space cross Bob’s type space) properly contains the set of full-support LPS’s on $S^b \times T^b$, and similarly with Ann and Bob interchanged. More loosely, a type structure is complete if it contains all possible full-support types and at least one non-full-support type (as per Section 2.4 above). Complete type structures exist for every finite game (Proposition 7.2). Figure 2.9 suggests that with this setup, we should be able to identify the IA strategies.

For $m \geq 0$, say there is *rationality and m th-order assumption of rationality* (RmAR) if conditions (a($m + 1$)) and (b($m + 1$)) of Section 2.2 hold. We have

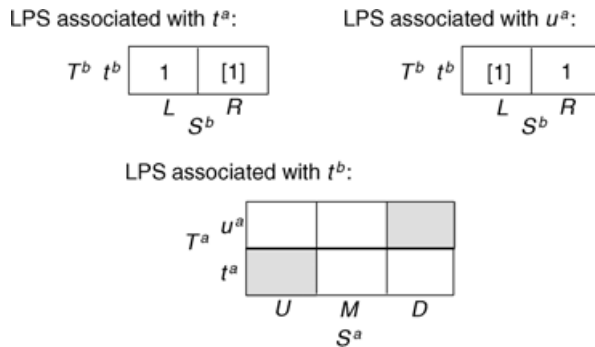


FIGURE 2.9.

(Theorem 9.1):

Start with a game and an associated complete type structure. Let $Q^a \times Q^b$ be the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying RmAR. Then $Q^a \times Q^b$ is the set of strategies that survive $(m + 1)$ rounds of IA.

2.7. A Negative Result

Note that our Theorem 9.1 actually identifies, for any m , the $(m + 1)$ -iteratively admissible strategies, not the IA strategies. Of course, for a given (finite) game, there is a number M such that for all $m \geq M$, the m -iteratively admissible strategies coincide with the IA strategies. Nevertheless, our result is not quite an epistemic condition for IA in all finite games. That would be one common condition—across all games—that yields IA. For example, one might hope to characterize the IA set as the projection of a set of states which is constructed in a uniform way in all complete type structures.

One would expect the RCAR set to be a natural candidate for this set of states. But the following negative result (Theorem 10.1) shows that RCAR will not work and is the reason for our limited statement of Theorem 9.1:

Start with a game in which Ann has more than one “strategically distinct” strategy, and an associated continuous complete type structure. Then no state satisfies RCAR.

For the meaning of a continuous type structure, see Definition 7.8. The complete type structure we get from our existence result (Proposition 7.2) is continuous.

In a certain sense, the result says that players cannot “reason all the way.” Here is an intuition for the result. Suppose the RCAR set is nonempty. Then there must be a type t^a for Ann that assumes each of the decreasing sequence of events (b1), (b2), ... (these events were defined in Section 2.2). That is, strategy–type pairs not in (b1) must be considered infinitely less likely than pairs in (b1); pairs not in (b2) must be considered infinitely less likely than pairs in (b2), and so on. Let $(\mu_0, \dots, \mu_{n-1})$ be the LPS associated with t^a . Figure 2.10 shows the most parsimonious way to arrange the measures μ_i , so that Ann indeed assumes each of (b1), (b2), ... But even in this case, we will run out of measures and Ann will not be able to assume any of the events (bn), (b(n + 1)), ... More loosely, at some point Ann will hit her primary hypothesis μ_0 , at which point there is no next (more likely) order of likelihood.

In the complete type structure we get from Proposition 7.2, each event (b(m + 1)) is “significantly” smaller than event (bm). This is because Bob has many types that assume the event (a(m – 1)) but not the event (am). So the measures μ_i do indeed have to be arranged as shown. This was not true in the incomplete structure of Figure 2.8, where these events do not shrink at all.

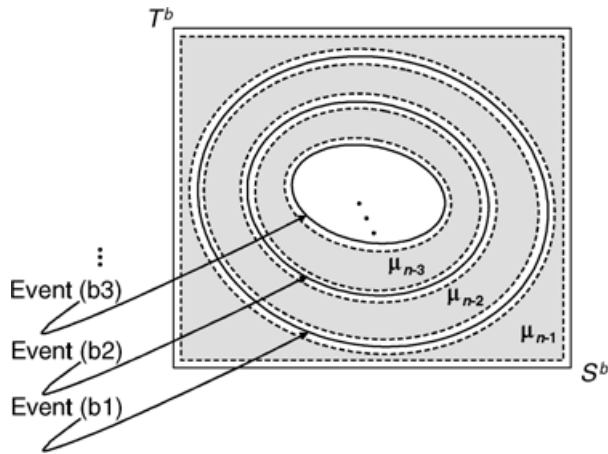


FIGURE 2.10.

That is why we had a state there satisfying RCAR, while no such state exists for the complete structure of Proposition 7.2.

2.8. The Ingredients

To recap: We begin with the fundamental inclusion–exclusion challenge identified by Samuelson (1992). Our resolution is to allow some states to be infinitely more likely than others. (We do this by using LPS’s and the concept of assumption.) Then we characterize the strategies consistent with RCAR as the SAS’s of a game. In a complete type structure, the strategies consistent with RmAR are those that survive $(m + 1)$ rounds of iterated admissibility. However, under certain conditions, RCAR in a complete structure is impossible.

Our examination of admissibility builds on fundamental work on the tree by Battigalli and Siniscalchi (2002). They study the solution concept of extensive-form rationalizability (EFR), an extensive-form analog to the iteratively undominated (IU) strategies. (The concept was defined by Pearce (1984) and later simplified by Battigalli (1997).)

B-S used conditional probability systems (CPS’s) to describe what players believe given what they observe in the tree. They next introduced the concept of strong belief. (This is the requirement that a player assign probability 1 to an event at each information set that is consistent with the event.) With rationality defined for the tree, they showed:

Start with a game and an associated (CPS-based) complete type structure. Let $Q^a \times Q^b$ be the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying rationality and m th-order strong belief of rationality. Then $Q^a \times Q^b$ is the set of strategies that survive $(m + 1)$ rounds of elimination of EFR.

Clearly, our Theorem 9.1 (previewed in Section 2.6) is closely related. In terms of ingredients, LPS's and CPS's can be formally related. See Halpern (2007) for a general treatment. Assumption can be viewed as a strategic-form analog to strong belief. Asheim and S¸ovik (2005) explored this connection; see also our companion piece (Brandenburger, Friedenberg, and Keisler (2006)). The role of completeness in our analysis is similar to its role in B-S.

There are also similarities in terms of output. IA and EFR are outcome equivalent in generic trees.⁵ Of course, many games of interest are non-generic.⁶ In simultaneous-move games, EFR reduces to IU. IA and EFR will then differ whenever IA and IU do.

There are other differences between the two analyses. In B-S, there is no analog to our negative result (Theorem 10.1). The reason is that full-support LPS's are, in a particular sense, more informative than CPS's on the tree. The online Supplemental material gives an exact treatment of this difference.

Also, we cover the case of incomplete type structures via our Theorem 8.1 (previewed in Section 2.5). We think of a particular incomplete structure as giving the "context" in which the game is played. In line with Savage's Small-Worlds idea in decision theory (Savage (1954, pp. 82–91)), who the players are in the given game can be seen as a shorthand for their experiences before the game. The players' possible characteristics—including their possible types—then reflect the prior history or context. (Seen in this light, complete structures represent a special "context-free" case, in which there has been no narrowing down of types.) SAS's are our characterization of the epistemic condition of RCAR in the contextual case.⁷

3. SAS'S AND THE IA SET

We now begin the formal treatment. Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$, where S^a, S^b are the (finite) strategy sets, and π^a, π^b are payoff functions for Ann and Bob, respectively.⁸ Given a finite set X , let $\mathcal{M}(X)$ denote the set of all probability measures on X . The definitions to come all have counterparts with a and b reversed. We extend π^a to $\mathcal{M}(S^a) \times \mathcal{M}(S^b)$ in the usual way: $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a, s^b) \in S^a \times S^b} \sigma^a(s^a) \sigma^b(s^b) \pi^a(s^a, s^b)$. Throughout, we adopt the convention that in a product $X \times Y$, if $X = \emptyset$, then $Y = \emptyset$ (and vice versa).

⁵See Brandenburger and Friedenberg (2003). Shimoji (2004) has a result relating IA and EFR, where EFR is defined relative to "normal-form information sets" (Mailath, Samuelson, and Swinkels (1993)).

⁶Examples include auction games, voting games, Bertrand, and zero-sum games. See Mertens (1989) and Marx and Swinkels (1997) for the same observation on nongenericity, and lists of examples.

⁷The online Supplemental material (Brandenburger, Friedenberg, and Keisler (2008)) contains discussion of other related work.

⁸For notational simplicity, we restrict attention throughout to two-player games. But the analysis can be extended without change to games with three or more players.

DEFINITION 3.1: Fix $X \times Y \subseteq S^a \times S^b$. A strategy $s^a \in X$ is *weakly dominated with respect to* $X \times Y$ if there exists $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y$ and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is *admissible with respect to* $X \times Y$. If s^a is admissible with respect to $S^a \times S^b$, simply say that s^a is *admissible*.

Write $\text{Supp } \sigma$ for the support of σ . We have the usual equivalence:

LEMMA 3.1: *A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there exists $\sigma^b \in \mathcal{M}(S^b)$, with $\text{Supp } \sigma^b = Y$, such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$.*

DEFINITION 3.2: Say r^a *supports* s^a if there exists some $\sigma^a \in \mathcal{M}(S^a)$ with $r^a \in \text{Supp } \sigma^a$ and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in S^b$. Write $\text{su}(s^a)$ for the set of $r^a \in S^a$ that support s^a .

In words, the strategy r^a is contained in $\text{su}(s^a)$ if it is part of a convex combination of Ann's strategies that is equivalent for her to s^a .

We can now define SAS's and the IA set:

DEFINITION 3.3: Fix $Q^a \times Q^b \subseteq S^a \times S^b$. The set $Q^a \times Q^b$ is a *self-admissible set (SAS)* if:

- (i) each $s^a \in Q^a$ is admissible;
- (ii) each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$;
- (iii) for any $s^a \in Q^a$, if $r^a \in \text{su}(s^a)$, then $r^a \in Q^a$.

Likewise for each $s^b \in Q^b$.

DEFINITION 3.4: Set $S_0^i = S^i$ for $i = a, b$ and define inductively

$$S_{m+1}^i = \{s^i \in S_m^i : s^i \text{ is admissible with respect to } S_m^a \times S_m^b\}.$$

A strategy $s^i \in S_m^i$ is called *m-admissible*. A strategy $s^i \in \bigcap_{m=0}^{\infty} S_m^i$ is called *iteratively admissible (IA)*.

Note that there is an M such that $\bigcap_{m=0}^{\infty} S_m^i = S_M^i$ for $i = a, b$. Moreover, each set S_m^i is nonempty and, hence, the IA set is nonempty.

4. LEXICOGRAPHIC PROBABILITY SYSTEMS

Given a Polish space Ω , it will be helpful to fix a metric. (So "Polish" will mean complete separable metric.) Let $\mathcal{M}(\Omega)$ be the space of Borel probability measures on Ω with the Prohorov metric. Recall that $\mathcal{M}(\Omega)$ is again a Polish space and has the topology of weak convergence (Billingsley (1968,

Appendix III)). Let $\mathcal{N}(\Omega)$ be the set of all finite sequences of Borel probability measures on Ω . That is, if $\sigma \in \mathcal{N}(\Omega)$, then there is some integer n with $\sigma = (\mu_0, \dots, \mu_{n-1})$.

Define a metric on $\mathcal{N}(\Omega)$ as follows. The distance between two sequences of measures $(\mu_0, \dots, \mu_{n-1})$ and $(\nu_0, \dots, \nu_{n-1})$ of the same length is the maximum of the Prohorov distances between μ_i and ν_i for $i < n$. The distance between two sequences of measures of different lengths is 1. For each fixed n , this metric on the set of sequences in $\mathcal{N}(\Omega)$ of length n is easily seen to be separable and complete, and thus Polish (this is the usual finite product metric). The whole space $\mathcal{N}(\Omega)$ is thus a countable union of Polish spaces at uniform distance 1 from each other. This shows that $\mathcal{N}(\Omega)$ itself is a Polish space.

DEFINITION 4.1: Fix $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{N}(\Omega)$, for some integer n . Say σ is a *lexicographic probability system (LPS)* if σ is *mutually singular*—that is, for each $i = 0, \dots, n - 1$, there are Borel sets U_i in Ω with $\mu_i(U_i) = 1$ and $\mu_i(U_j) = 0$ for $i \neq j$. Write $\mathcal{L}(\Omega)$ for the set of LPS's and write $\overline{\mathcal{L}}(\Omega)$ for the closure of $\mathcal{L}(\Omega)$ in $\mathcal{N}(\Omega)$.

An LPS is a finite measure sequence where the measures are nonoverlapping (mutually singular). This has the usual interpretation: the player's primary hypothesis, secondary hypothesis, and so on, until an n th hypothesis.

In general, an LPS may have some null states which remain outside the support of each of its measures. We are also interested in the case that there are no such null states:

DEFINITION 4.2: A *full-support sequence* is a sequence $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{N}(\Omega)$ such that $\Omega = \bigcup_{i < n} \text{Supp } \mu_i$. We write $\mathcal{N}^+(\Omega)$ for the set of full-support sequences and write $\mathcal{L}^+(\Omega)$ for the set of *full-support LPS's*.

Here, $\text{Supp } \mu_i$ denotes the support of μ_i , that is, the smallest closed set with μ_i -measure 1. The space $\overline{\mathcal{L}}(\Omega)$ is Polish, since it is a closed subspace of the Polish space $\mathcal{N}(\Omega)$. Also, the sets $\mathcal{N}^+(\Omega)$, $\mathcal{L}(\Omega)$, and $\mathcal{L}^+(\Omega)$ are Borel (Corollary C.1).

Our definition of an LPS is an infinite version of the definition for finite spaces introduced by Blume, Brandenburger, and Dekel (henceforth BBD) (1991a). Infinite spaces play a crucial role in this paper—complete type structures (recall the discussion in Section 2.6) are infinite. (A note on terminology: BBD used the term LPS even when mutual singularity does not hold.)

5. ASSUMPTION

Here, we define formally the concept of assumption, which was introduced informally in Section 2.1. Fix a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1})$ for Ann and fix an event E . Intuitively, Ann assumes E if she considers E infinitely

more likely than not- E under σ . So, to define assumption, we first need to understand the idea of “infinitely more likely than.”

BBD (1991a, Definition 5.1) gave a definition of “infinitely more likely than” for the case of a finite space Ω and a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1})$ (see their Axiom 5’). They say that a point ω_1 is infinitely more likely than a point ω_2 if ω_1 comes before ω_2 in the lexicographic ordering. For disjoint events F and G , they require that F is nonempty and each point in F is infinitely more likely than each point in G . Formally, the requirement is that: F is nonempty, and for each $\omega_1 \in F$ and $\omega_2 \in G$, $\mu_j(\omega_1) > 0$ and $\mu_k(\omega_2) > 0$ implies $j < k$. (The same idea of “infinitely more likely than” can be found in Battigalli (1996, p. 186) and Asheim and Dufwenberg (2003).)

We want a general (i.e., infinite) analog to this definition, so we work with open sets rather than just points. Call F_0 a *part* of F if $F_0 = U \cap F \neq \emptyset$ for some open U . Instead of asking that each point in F be infinitely more likely than each point in G , we require that each part of F be infinitely more likely than each part of G .

DEFINITION 5.1: Fix a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(\Omega)$ and disjoint events F and G . Then F is *infinitely more likely than* G under σ if F is nonempty and, for any part F_0 of F :

- (a) $\mu_i(F_0) > 0$ for some i ;
- (b) if $\mu_j(F_0) > 0$ and there is a part G_0 of G with $\mu_k(G_0) > 0$, then $j < k$.

Note that for finite Ω , this is equivalent to the BBD definition. In particular, condition (a) is then automatically satisfied (since every point gets positive probability under some μ_i). In the general case, we need to require (a) explicitly. Without it, we could have that F is infinitely more likely than G , but at the same time G is infinitely more likely than F . This would not make sense. (See the online Supplemental material.)

The idea of assumption of an event E is simply that E is considered infinitely more likely than not- E :

DEFINITION 5.2: Fix an event E and a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(\Omega)$. Say E is *assumed* under σ if E is infinitely more likely than $\Omega \setminus E$ under σ .

We have the following characterization of assumption⁹:

PROPOSITION 5.1: Fix an event E and a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(\Omega)$. An event E is assumed under σ if and only if there is a j such that:

- (i) $\mu_i(E) = 1$ for all $i \leq j$;
- (ii) $\mu_i(E) = 0$ for all $i > j$;

⁹Proofs not given in the main text can be found in the [Appendices](#).

(iii) if U is open with $U \cap E \neq \emptyset$, then $\mu_i(U \cap E) > 0$ for some i .

(We will sometimes say that E is *assumed at level j* . Also, we will refer to conditions (i)–(iii) of Proposition 5.1 as conditions (i)–(iii) of assumption.)

Note that if Ω is finite, conditions (i) and (ii) imply condition (iii). But this is not the case when Ω is infinite. (See the online Supplemental material.)

As with the usual notion of “belief” of an event E , assumption can be given an axiomatic treatment. Appendix A proposes two axioms: *Strict Determination* says that whenever Ann strictly prefers one act to another conditional on E , she has the same preference unconditionally. *Nontriviality* says that, conditional on any part of E , she can have a strict preference. In Appendix A, we show that Ann assumes E if and only if her preferences satisfy these axioms. We also relate this axiomatization to the axiomatization of “infinitely more likely than” in BBD (1991a, Definition 5.1).

6. PROPERTIES OF ASSUMPTION

We next mention some properties of assumption. (Again, we use an overbar to denote closure.)

PROPERTY 6.1—Convexity: *If E and F are assumed under σ at level j , then any Borel set G lying between $E \cap F$ and $E \cup F$ is also assumed under σ at level j .*

PROPERTY 6.2—Closure: *If E and F are assumed under σ at level j , then $\overline{E} = \overline{F}$. If E and F are assumed under σ , then either $\overline{E} \subseteq \overline{F}$ or $\overline{F} \subseteq \overline{E}$.*

The Convexity property refers to convexity in the sense of orderings (where the order is set inclusion), and is a two-sided monotonicity. The Closure property implies that, for a finite space, there is only one set that is assumed at each level. Also, in the finite case, if E and F are both assumed, then $\overline{E} \subseteq \overline{F}$ or $\overline{F} \subseteq \overline{E}$. Neither statement is true for an infinite space.

Overall, the mental picture we suggest for assumption is of rungs of a ladder, separated by gaps, where each rung is a convex family of sets with the same closure. (Each rung corresponds to the events assumed at the particular level.)

Next, notice that assumption is not monotonic. Here is an example: Set $\Omega = [0, 1] \cup \{2, 3\}$, and let $\sigma = (\mu_0, \mu_1)$ be a full-support LPS where μ_0 is uniform on $[0, 1]$ and $\mu_1(\{2\}) = \mu_1(\{3\}) = \frac{1}{2}$. Then σ assumes $(0, 1]$ but not $(0, 1] \cup \{2\}$.

The best way to understand this nonmonotonicity is in terms of our axiomatic treatment.¹⁰ Suppose Ann assumes $(0, 1]$ —that is, when she has a strict preference, she is willing to make a decision based solely on $(0, 1]$. (This is Strict Determination.) It does not seem natural to require that Ann also be willing to make a decision based only on $(0, 1] \cup \{2\}$. After all, she considers the

¹⁰We thank a referee for this line of argument.

possibility that 2 obtains. (Nontriviality implies that the state 2 must get positive weight under some measure—as it does under μ_1 .) Once she considers this possibility, presumably she should also consider the possibility that 3 obtains. (To give 2 positive probability, she must look to her secondary hypothesis, which also gives 3 positive probability.) Of course, the state 3 may well matter for her preferences.

On the other hand, if Ann assumes $(0, 1]$, then certainly she should assume $[0, 1]$. Admitting the possibility of 0 does not force her to look to her secondary hypothesis—it does not force her to consider 2 or 3 possible. Formally, Ann assumes $[0, 1]$ and $(0, 1]$ at the same level. Convexity then requires her to assume $[0, 1]$ (at the same level).

Because of the nonmonotonicity, assumption fails one direction of conjunction. Returning to the example, Ann assumes $(0, 1] \cap ((0, 1] \cup \{2\})$ even though she does not assume $(0, 1] \cup \{2\}$. But the other direction of conjunction, and the analog for disjunction, are satisfied:

PROPERTY 6.3—Conjunction and Disjunction: *Fix Borel sets E_1, E_2, \dots in Ω and suppose that, for each m , E_m is assumed under σ . Then $\bigcap_m E_m$ and $\bigcup_m E_m$ are assumed under σ .*

7. TYPE STRUCTURES

Fix again a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$.

DEFINITION 7.1: An (S^a, S^b) -based type structure is a structure

$$\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle,$$

where T^a and T^b are nonempty Polish spaces, and $\lambda^a: T^a \rightarrow \overline{\mathcal{L}}(S^b \times T^b)$ and $\lambda^b: T^b \rightarrow \overline{\mathcal{L}}(S^a \times T^a)$ are Borel measurable. Members of T^a, T^b are called *types*. Members of $S^a \times T^a \times S^b \times T^b$ are called *states (of the world)*. A type structure is called *lexicographic* if $\lambda^a: T^a \rightarrow \mathcal{L}(S^b \times T^b)$ and $\lambda^b: T^b \rightarrow \mathcal{L}(S^a \times T^a)$.

Definition 7.1 is based on a standard epistemic definition: A type structure enriches the basic description of a game by appending spaces of epistemic types for both players, where a type for a player is associated with a sequence of measures on the strategies and types for the other player. The difference from the standard definition is the use of a sequence of measures rather than one measure.

Our primary focus will be on lexicographic type structures, which have a natural interpretation in a game setting. Nonlexicographic type structures will play a useful role in the construction of lexicographic type structures. Note that lexicographic type structures can contain two different kinds of types—those associated with full-support LPS's and those associated with non-full-support LPS's. The reason for this was discussed in Section 2.4.

The following definitions apply to a given game and type structure. As before, they also have counterparts with a and b reversed. Write $\text{marg}_{S^b} \mu_i$ for the marginal on S^b of the measure μ_i .

DEFINITION 7.2: A strategy s^a is *optimal* under $\sigma = (\mu_0, \dots, \mu_{n-1})$ if $\sigma \in \mathcal{L}(S^b \times T^b)$ and

$$\left(\pi^a(s^a, \text{marg}_{S^b} \mu_i(s^b))\right)_{i=0}^{n-1} \geq^L \left(\pi^a(r^a, \text{marg}_{S^b} \mu_i(s^b))\right)_{i=0}^{n-1}$$

for all $r^a \in S^a$.¹¹

In words, Ann will prefer strategy s^a to strategy r^a if the associated sequence of expected payoffs under s^a is lexicographically greater than the sequence under r^a . (If σ is a length-1 LPS (μ_0) , we will sometimes say that s^a is optimal under the measure μ_0 if it is optimal under (μ_0) .)

DEFINITION 7.3: A type $t^a \in T^a$ has *full support* if $\lambda^a(t^a)$ is a full-support LPS.

DEFINITION 7.4: A strategy–type pair $(s^a, t^a) \in S^a \times T^a$ is *rational* if t^a has full support and s^a is optimal under $\lambda^a(t^a)$.

This is the usual definition of rationality, plus the full-support requirement, which is to capture our basic admissibility requirement. The following two lemmas say this formally:

LEMMA 7.1—BBD (1991b): *Suppose s^a is optimal under a full-support LPS $(\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(S^b \times T^b)$. Then there is a length-1 full-support LPS $(\nu_0) \in \mathcal{L}^+(S^b \times T^b)$, under which s^a is optimal.*

Together with Lemma 3.1, this gives the following lemma:

LEMMA 7.2: *If (s^a, t^a) is rational, then s^a is admissible.*

Fix an event $E \subseteq S^b \times T^b$ and write

$$A^a(E) = \{t^a \in T^a : \lambda^a(t^a) \text{ assumes } E\}.$$

The set $A^a(E)$ is Borel (Lemma C.3).

Let R_1^a be the set of rational strategy–type pairs (s^a, t^a) . For finite m , define R_m^a inductively by

$$R_{m+1}^a = R_m^a \cap [S^a \times A^a(R_m^b)].$$

¹¹If $x = (x_0, \dots, x_{n-1})$ and $y = (y_0, \dots, y_{n-1})$, then $x \geq^L y$ if and only if $y_j > x_j$ implies $x_k > y_k$ for some $k < j$.

The sets R_m^a are Borel (Lemma C.4).

DEFINITION 7.5: If $(s^a, t^a, s^b, t^b) \in R_{m+1}^a \times R_{m+1}^b$, say there is *rationality and m th-order assumption of rationality (RmAR)* at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^\infty R_m^a \times \bigcap_{m=1}^\infty R_m^b$, say there is *rationality and common assumption of rationality (RCAR)* at this state.

In words, there is RCAR at a state if Ann is rational, Ann assumes the event “Bob is rational,” Ann assumes the event “Bob is rational and assumes Ann is rational,” and so on, and similarly starting with Bob.

Note, we cannot replace this definition with $\widehat{R}_1^a = R_1^a$ and $\widehat{R}_{m+1}^a = \widehat{R}_1^a \cap [S^a \times A^a(\widehat{R}_m^b)]$. To clarify, suppose $(s^a, t^a) \in R_3^a$. Then $(s^a, t^a) \in R_1^a \cap [S^a \times A^a(R_1^b)] \cap [S^a \times A^a(R_1^b \cap [S^b \times A^b(R_1^a)])]$. In words, Ann is rational, she assumes the event “Bob is rational,” and she assumes the event “Bob is rational and assumes Ann is rational.” Now suppose $(s^a, t^a) \in \widehat{R}_3^a$. Then $(s^a, t^a) \in R_1^a \cap [S^a \times A^a(R_1^b \cap [S^b \times A^b(R_1^a)])]$. In words, Ann is rational, and she assumes the event “Bob is rational and assumes Ann is rational.” But, because assumption is not monotonic, she might not assume the event “Bob is rational.” We think that under a good definition of R2AR, Ann should assume this event.

Next is a notion of equivalence between type structures.

DEFINITION 7.6: Two type structures $\langle S^a, S^b, T^a, T^b, \kappa^a, \kappa^b \rangle$ and $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ are *equivalent* if:

- (i) they have the same strategy and type spaces;
- (ii) for each $t^a \in T^a$, if either $\kappa^a(t^a)$ or $\lambda^a(t^a)$ belongs to $\mathcal{L}^+(S^b \times T^b)$, then $\kappa^a(t^a) = \lambda^a(t^a)$ (and likewise with a and b reversed).

PROPOSITION 7.1:

- (i) For every type structure there is an equivalent lexicographic type structure.
- (ii) If two type structures are equivalent, then for each m they have the same R_m^a and R_m^b sets.

This proposition shows that any statement about rationality and m th-order assumption of rationality (for any m) that is true for every lexicographic type structure is true for every type structure. Conceptually, we are interested in type structures which satisfy the hypothesis of being lexicographic, but the proposition tells us that we will never need this hypothesis in our theorems. In practice, then, we will state and prove theorems for arbitrary type structures. By Proposition 7.1, in these proofs we can always assume without loss of generality that the type structure is lexicographic.

We conclude this section with the idea of a complete type structure (adapted from Brandenburger (2003)).

DEFINITION 7.7: A type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ is *complete* if $\mathcal{L}^+(S^b \times T^b) \subseteq_{\neq} \text{range } \lambda^a$ and $\mathcal{L}^+(S^a \times T^a) \subseteq_{\neq} \text{range } \lambda^b$.

In words, a complete structure contains all full-support LPS's for Ann and Bob, and (at least) one non-full-support LPS.¹² (Refer back to Sections 2.4 and 2.6.) We see at once from the definition that any type structure which is equivalent to a complete type structure is complete.

PROPOSITION 7.2: *For any finite sets S^a and S^b , there is a complete type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ such that the maps λ^a and λ^b are continuous.*

DEFINITION 7.8: A type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ is *continuous* if it is equivalent to a type structure where the λ^a and λ^b maps are continuous.

Thus, in a continuous type structure, players associate neighboring full-support LPS's with neighboring full-support types. Propositions 7.1 and 7.2 immediately give the following corollary:

COROLLARY 7.1: *For any finite sets S^a and S^b , there exists a complete continuous lexicographic (S^a, S^b) -based type structure.*

8. CHARACTERIZATION OF RCAR

THEOREM 8.1:

(i) *Fix a type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $\text{proj}_{S^a} \bigcap_{m=1}^\infty R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^\infty R_m^b$ is an SAS.*

(ii) *Fix an SAS $Q^a \times Q^b$. There is a lexicographic type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^\infty R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^\infty R_m^b$.*

PROOF: For part (i), if $\bigcap_m R_m^a \times \bigcap_m R_m^b = \emptyset$, then the conditions of an SAS are automatically satisfied. So we will suppose this set is nonempty.

Fix $s^a \in \text{proj}_{S^a} \bigcap_m R_m^a$. Then $(s^a, t^a) \in \bigcap_m R_m^a$ for some $t^a \in T^a$. Certainly $(s^a, t^a) \in R_1^a$. Using Lemma 7.2, s^a is admissible, establishing condition (i) of an SAS. By Property 6.3, $t^a \in A^a(\bigcap_m R_m^b)$. We therefore get a picture like Figure 8.1 (for some $j < n$), and, as illustrated,

$$\bigcup_{i \leq j} \text{Supp marg}_{S^b} \mu_i = \text{proj}_{S^b} \bigcap_m R_m^b.$$

(This is formally established as Lemma D.1 and uses condition (iii) of assumption.) As in Lemma 7.1, there is a length-1 LPS (ν_0) on S^b , with $\text{Supp } \nu_0 = \text{proj}_{S^b} \bigcap_m R_m^b$, under which s^a is optimal. Thus s^a is admissible with respect

¹²In the literature, the more common concept of a model of all possible types is the universal (or canonical) model. (See Armbruster and Böge (1979), Böge and Eisele (1979), Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993), and Battigalli and Siniscalchi (1999), among others.) The completeness concept is well suited to our analysis.

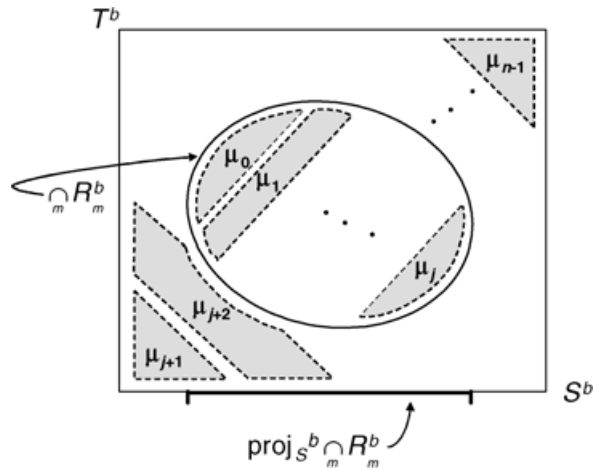


FIGURE 8.1.

to $S^a \times \text{proj}_{S^b} \cap_m R_m^b$, establishing condition (ii) of an SAS. Next suppose $r^a \in \text{su}(s^a)$. Then, for any t^a , $(s^a, t^a) \in R_1^a$ implies $(r^a, t^a) \in R_1^a$ (Lemma D.2), and so we have for all m , $(s^a, t^a) \in R_m^a$ implies $(r^a, t^a) \in R_m^a$. This establishes condition (iii) of an SAS.

For part (ii) of the theorem, fix an SAS $Q^a \times Q^b$. (Recall the convention that if $Q^a = \emptyset$, then $Q^b = \emptyset$ and vice versa.) By conditions (i) and (ii) of an SAS, for each $s^a \in Q^a$ there are measures $\nu_0, \nu_1 \in \mathcal{M}(S^b)$, with $\text{Supp } \nu_0 = S^b$ and $\text{Supp } \nu_1 = Q^b$, under which s^a is optimal. We can choose ν_0 so that r^a is optimal under ν_0 if and only if $r^a \in \text{su}(s^a)$. (This is Lemma D.4.)

Define type spaces $T^a = Q^a \cup \{t_*^a\}$ and $T^b = Q^b \cup \{t_*^b\}$, where t_*^a and t_*^b are arbitrary labels. For $t^a = s^a \in Q^a$, the associated $\lambda^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$ will be a two-level full-support LPS (μ_0, μ_1) , where $\text{marg}_{S^b} \mu_0 = \nu_1$ and $\text{marg}_{S^b} \mu_1 = \nu_0$.¹³ (Further conditions are specified below.) Let $\lambda^a(t_*^a)$ be an element of $\mathcal{L}(S^b \times T^b) \setminus \mathcal{L}^+(S^b \times T^b)$. Define the map λ^b similarly.

Figure 8.2 shows the construction of $\lambda^a(t^a)$: Under the above specifications, points (s^b, s^b) on the diagonal are rational. That is, these points lie in R_1^b . Other points (r^b, s^b) are rational if and only if $r^b \in \text{su}(s^b)$. By condition (iii) of an SAS, $\text{su}(s^b) \subseteq Q^b$. So the set R_1^b contains the diagonal and is contained in the rectangle $Q^b \times Q^b$. Moreover, for each $s^b \in S^b$, $(s^b, t_*^b) \in (S^b \times T^b) \setminus R_1^b$. Thus we can take the measures μ_0 and μ_1 to satisfy

$$\begin{aligned} \text{marg}_{S^b} \mu_0 &= \nu_1, & \text{Supp } \mu_0 &= R_1^b, \\ \text{marg}_{S^b} \mu_1 &= \nu_0, & \text{Supp } \mu_1 &= (S^b \times T^b) \setminus R_1^b. \end{aligned}$$

Likewise for the map λ^b .

¹³We reverse the indices for consistency with the proof of Theorem 9.1 below.

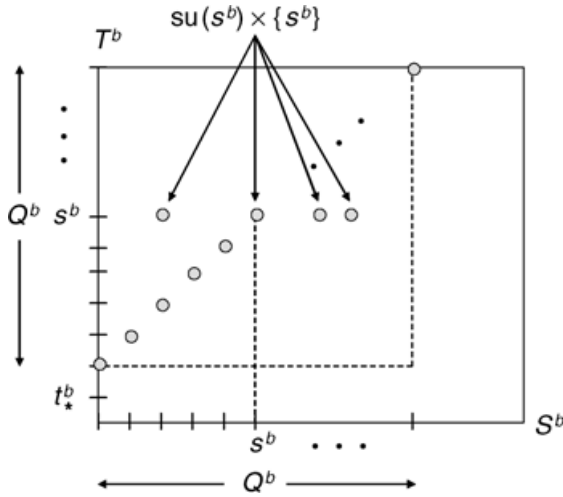


FIGURE 8.2.

We now show that $\text{proj}_{S^a} \bigcap_m R_m^a = Q^a$ and likewise for b . By the same argument as in the previous paragraph, $\text{proj}_{S^a} R_1^a = Q^a$. Moreover, each $t^a \in Q^a$ assumes R_1^b . (Conditions (i) and (ii) are immediate for $j = 0$. Condition (iii) follows immediately from the fact that $S^b \times T^b$ is finite and each $t^a \in Q^a$ has full support.) So $R_2^a = R_1^a$. Likewise for b . Thus $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m , by induction. Certainly $\text{proj}_{S^a} R_1^a \times \text{proj}_{S^b} R_1^b = Q^a \times Q^b$. It follows that $\text{proj}_{S^a} \bigcap_m R_m^a \times \text{proj}_{S^b} \bigcap_m R_m^b = Q^a \times Q^b$, as required. Q.E.D.

9. CHARACTERIZATION OF RmAR IN A COMPLETE STRUCTURE

THEOREM 9.1: Fix a complete type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, for each m ,

$$\text{proj}_{S^a} R_m^a \times \text{proj}_{S^b} R_m^b = S_m^a \times S_m^b.$$

PROOF: We may assume that the type structure is lexicographic. The proof is by induction on m . Begin by fixing some $(s^a, t^a) \in R_1^a$. By Lemma 7.2, $s^a \in S_1^a$. This shows that $\text{proj}_{S^a} R_1^a \times \text{proj}_{S^b} R_1^b \subseteq S_1^a \times S_1^b$.

Next fix some $s^a \in S_1^a$. By Lemma 3.1, there is an LPS $(\nu_0) \in \mathcal{L}^+(S^b)$ under which s^a is optimal. We want to construct an LPS $(\mu_0) \in \mathcal{L}^+(S^b \times T^b)$ with $\text{marg}_{S^b} \mu_0 = \nu_0$. By completeness, there will then be a type t^a with $\lambda^a(t^a) = (\mu_0)$. By construction, the pair $(s^a, t^a) \in R_1^a$. This will establish that $\text{proj}_{S^a} R_1^a \times \text{proj}_{S^b} R_1^b = S_1^a \times S_1^b$.

To construct (μ_0) , fix some $s^b \in S^b$ and set $X = \{s^b\} \times T^b$. Note that $\nu_0(s^b) > 0$. By rescaling and combining measures over different s^b , it is enough to find

$(\xi_0) \in \mathcal{L}^+(X)$. By separability, X has a countable dense subset Y . So by assigning positive weight to each point in Y , we get a measure ξ_0 , where $\xi_0(Y) = 1$ and $\text{Supp } \xi_0$ is the closure of Y , as required.

Now assume the result for all $1 \leq i \leq m$. We will show it is also true for $i = m + 1$. Fix some $(s^a, t^a) \in R_{m+1}^a$, where $\lambda^a(t^a) = (\mu_0, \dots, \mu_{n-1})$. Then $(s^a, t^a) \in R_m^a$ and so, by the induction hypothesis, $s^a \in S_m^a$. Also, $t^a \in A^a(R_m^b)$. Since $\text{proj}_{S^b} R_m^b = S_m^b$, by the induction hypothesis, we get a picture like Figure 9.1 (for some $j < n$). By the same argument as in the proof of Theorem 8.1, we conclude that s^a is admissible with respect to $S^a \times S_m^b$ (so certainly with respect to $S_m^a \times S_m^b$). Thus $s^a \in S_{m+1}^a$.

Next fix some $s^a \in S_{m+1}^a$. It will be useful to set $S_0^b = S^b$ and $R_0^b = S^b \times T^b$. For each $0 \leq i \leq m$, there is a measure $\nu_i \in \mathcal{M}(S^b)$, with $\text{Supp } \nu_i = S_i^b$, under which s^a is optimal among all strategies in S^a . (This is Lemma E.1, which uses Lemma 3.1.) Thus s^a is (lexicographically) optimal under the sequence of measures (ν_0, \dots, ν_m) . Also, using the induction hypothesis, $S_i^b = \text{proj}_{S^b} R_i^b$ for all $0 \leq i \leq m$. We want to construct an LPS $(\mu_0, \dots, \mu_m) \in \mathcal{L}^+(S^b \times T^b)$ where:

- (i) $\text{marg}_{S^b} \mu_i = \nu_{m-i}$;
- (ii) R_i^b is assumed at level $m - i$.

It will then follow from completeness that there is a t^a with $\lambda^a(t^a) = (\mu_0, \dots, \mu_m)$, and hence $(s^a, t^a) \in R_{m+1}^a$. (Refer to Figure 9.2.)

Now fix some $s^b \in S^b$ and set $X = \{s^b\} \times T^b$ as above. Let h be the greatest $i \leq m$ such that $s^b \in S_i^b$. Note that for each $i \leq h$ we have $s^b \in S_i^b = \text{Supp } \nu_i$ and so $\nu_i(s^b) > 0$.

By rescaling and combining the measures over different s^b , it is enough (using Lemma B.1) to find $(\xi_0, \dots, \xi_h) \in \mathcal{L}^+(X)$ with:

- (iii) $\xi_0(X \cap R_h^b) = 1$;
- (iv) $\xi_i(X \cap (R_{h-i}^b \setminus R_{h-i+1}^b)) = 1$ for each $1 \leq i \leq h$;

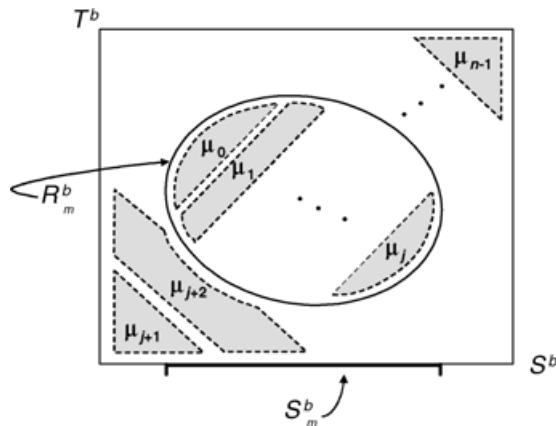


FIGURE 9.1.

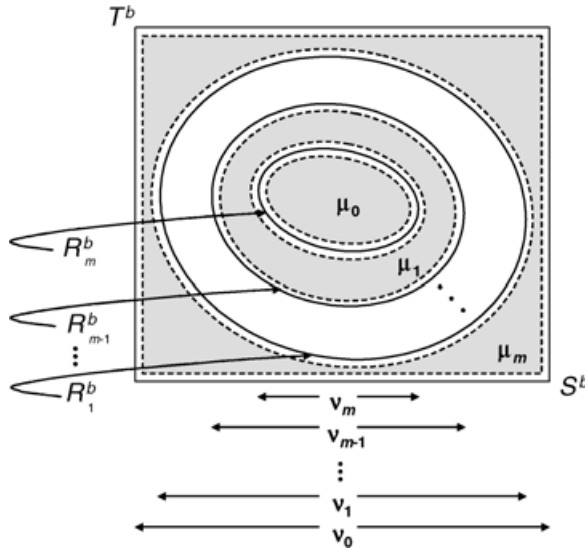


FIGURE 9.2.

(v) $X \cap R_{h-i}^b \subseteq \bigcup_{j=0}^i \text{Supp } \xi_j$ for each $0 \leq i \leq h$.

Each R_{h-i}^b is Borel (Lemma C.4). We also have $\text{proj}_{S^b} R_{h-i}^b = \text{proj}_{S^b} (R_{h-i}^b \setminus R_{h-i+1}^b)$. (This is Lemma E.3. It is the place where we use the fact that a complete lexicographic type structure has a non-full-support LPS.) Since $s^b \in \text{proj}_{S^b} R_{h-i}^b$, for each $1 \leq i \leq h$ the set $X_i = X \cap (R_{h-i}^b \setminus R_{h-i+1}^b)$ is nonempty. The set $X_0 = X \cap R_h^b$ is also nonempty. The proof is finished by the same argument as in the base step above: By separability, each X_i has a countable dense subset Y_i . Assign positive probability to each point in Y_i to get a measure ξ_i , where $\xi_i(Y_i) = 1$ and $\text{Supp } \xi_i$ is the closure of Y_i . Then $(\xi_0, \dots, \xi_h) \in \mathcal{L}^+(X)$ and satisfies (iii)–(v), completing the induction. *Q.E.D.*

10. A NEGATIVE RESULT

DEFINITION 10.1: Say that player a is *indifferent* if $\pi^a(r^a, s^b) = \pi^a(s^a, s^b)$ for all r^a, s^a, s^b .

So if a player is not indifferent, then he has more than one “strategically distinct” strategy.

THEOREM 10.1: *Fix a complete continuous type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. If player a is not indifferent, then there is no state at which there is RCAR. In fact,*

$$\bigcap_{m=1}^{\infty} R_m^a = \bigcap_{m=1}^{\infty} R_m^b = \emptyset.$$

We come back to Theorem 10.1 in Sections 11D and 11E.

11. DISCUSSION

Here, we discuss some open questions. The online Supplemental material contains other conceptual and technical discussion.

A. LPS's: We define an LPS to be a finite sequence of probability measures, not an infinite sequence. The main reason is that finite sequences suffice for what we do. But it would certainly be worth exploring extensions of our definition (see Halpern (2007)).

Would Theorem 10.1 go through with infinite sequences of measures? The intuition given in Section 2.7 appears to depend only on the condition that an LPS has a primary hypothesis, secondary hypothesis, and so forth. Given this, we will eventually hit the primary hypothesis, when trying to “count on” smaller and smaller events. In other words, it seems that the well-foundedness of an LPS is really what is responsible for the impossibility. The idea that a player has an initial hypothesis about a game seems very basic. That said, we do not know if Theorem 10.1 would be overturned if we used non-well-founded LPS's.

B. Assumption: A weaker concept than assumption of an event E is to require only belief at level 0. That is, given an LPS $(\mu_0, \dots, \mu_{n-1})$, we ask only that $\mu_0(E) = 1$. This is the concept used in Brandenburger (1992) and (effectively) in Börgers (1994). Ben Porath (1997) studied an extensive-form analog (which is, accordingly, weaker than strong belief).

All three papers obtain $S^\infty W$ strategies. (This is the Dekel and Fudenberg (1990) concept of one round of deletion of inadmissible strategies followed by iterated deletion of strongly dominated strategies.) Let us recast the analysis in the current epistemic framework.

Call a subset $Q^a \times Q^b$ of $S^a \times S^b$ a *weak best-response set* (WBRS) if (i) each $s^a \in Q^a$ is admissible, (ii) each $s^a \in Q^a$ is not strongly dominated with respect to $S^a \times Q^b$; and likewise with a and b interchanged. Every WBRS is contained in the $S^\infty W$ set, and the $S^\infty W$ set is a WBRS. We have the following analog to

our Theorem 8.1:

Let $Q^a \times Q^b$ be the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying rationality and common belief at level 0 of rationality. Then $Q^a \times Q^b$ is a WBRS. Conversely, given a WBRS $Q^a \times Q^b$, there is a type structure such that $Q^a \times Q^b$ is contained in the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying rationality and common belief at level 0 of rationality.

(Note that here the converse only has inclusion not equality.) We are not aware of an analog to Theorem 9.1.

C. Irrationality: We noted in Section 2.4 that there are two forms of irrationality in the paper: strategy–type pairs (s^a, t^a) where s^a is not optimal under $\lambda^a(t^a)$, and strategy–type pairs (s^a, t^a) where $\lambda^a(t^a)$ is not full support. The presence of a non-full-support type is needed in the proofs of each of our three main theorems (Theorems 8.1, 9.1, and 10.1). In each case, the key fact is that there is a type t^a so that each (s^a, t^a) is irrational.

This raises the question: What would happen if we required all types to have full support—that is, if we ruled out the second form of irrationality? The strategies played under RCAR would still constitute an SAS (Theorem 8.1(i)). But, as the discussion of Figure 2.6 showed, not every SAS could now arise under RCAR. We do not know what subfamily of the SAS’s would result and we leave this as an open question.

D. Continuity: In a continuous structure, players associate neighboring full-support LPS’s with neighboring full-support types (Definition 7.8). Theorem 10.1 made use of this condition in addition to the condition that Ann is not indifferent. Under these hypotheses, $S^a \times T^a$ contains a non-empty open set of irrational pairs. This is used to get the first step of an induction (Lemma F.1). At each later step of the induction, continuity is again needed to guarantee that the pre-image of an open set is still open.

What happens to Theorem 10.1 if continuity is dropped? Alternatively put, does there exist a complete type structure in which the RCAR set is nonempty? We do not know.

E. Infinite Games: Finally, Theorem 10.1 may be suggestive of limitations to the analysis of infinite games.¹⁴ For a fixed infinite game, it may be that one needs the full force of RCAR in a complete structure to obtain IA. Will this be possible? Of course, to answer this question, we have to rebuild all the ingredients of this paper for infinite games. This must be left to future work.

¹⁴We are grateful to Eddie Dekel for this observation.

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APPENDIX A: PREFERENCE BASIS

We begin with an axiomatic justification of assumption, that is, the conditions (i)–(iii) of Proposition 5.1.

Let Ω be a Polish space and let \mathcal{A} be the set of all measurable functions from Ω to $[0, 1]$. A particular function $x \in \mathcal{A}$ is an *act*, where $x(\omega)$ is the payoff to the player of choosing the act x if the true state is $\omega \in \Omega$. For $x, y \in \mathcal{A}$ and $0 \leq \alpha \leq 1$, write $\alpha x + (1 - \alpha)y$ for the act that in state ω gives payoff $\alpha x(\omega) + (1 - \alpha)y(\omega)$. For $c \in [0, 1]$, write \vec{c} for the constant act associated with c ; that is, $\vec{c}(\omega) = c$ for all $\omega \in \Omega$. Also, given acts $x, z \in \mathcal{A}$ and a Borel subset E in Ω , write $(x_E, z_{\Omega \setminus E})$ for the act:

$$(x_E, z_{\Omega \setminus E})(\omega) = \begin{cases} x(\omega), & \text{if } \omega \in E, \\ z(\omega), & \text{if } \omega \notin E. \end{cases}$$

Let \succsim be a preference relation on \mathcal{A} and write \succ (resp. \sim) for strict preference (resp. indifference). We maintain two axioms throughout:

A1—Order: \succsim is a complete, transitive, reflexive binary relation on \mathcal{A} .

A2—Independence: For all $x, y, z \in \mathcal{A}$ and $0 < \alpha \leq 1$,

$$x \succ y \text{ implies } \alpha x + (1 - \alpha)z \succ \alpha y + (1 - \alpha)z \quad \text{and}$$

$$x \sim y \text{ implies } \alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z.$$

Given a Borel set E , define conditional preference given E in the usual way:

DEFINITION A.1: $x \succsim_E y$ if for some $z \in \mathcal{A}$, $(x_E, z_{\Omega \setminus E}) \succsim (y_E, z_{\Omega \setminus E})$.

(As is well known, under A1 and A2, $(x_E, z_{\Omega \setminus E}) \succ (y_E, z_{\Omega \setminus E})$ holds for all z if it holds for some z .)

Given a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(\Omega)$, define \succsim^σ on \mathcal{A} by

$$x \succsim^\sigma y \iff \left(\int_{\Omega} x(\omega) d\mu_i(\omega) \right)_{i=0}^{n-1} \geq^L \left(\int_{\Omega} y(\omega) d\mu_i(\omega) \right)_{i=0}^{n-1}.$$

DEFINITION A.2: Say a set E is *believed* under \succsim if E is Borel and, for all $x, y \in \mathcal{A}$, $x \sim_{\Omega \setminus E} y$.

This is just the statement that the event $\Omega \setminus E$ is Savage-null. We have the following characterization of belief.

PROPOSITION A.1: Fix $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(\Omega)$ and a Borel set E in Ω . The following statements are equivalent:

- (i) $\mu_i(E) = 1$ for all i .
- (ii) E is believed under \succsim^σ .

PROOF: Suppose (i) holds. Then $\mu_i(\Omega \setminus E) = 0$ for all i , and so for any $x, y \in \mathcal{A}$, $x \sim_{\Omega \setminus E}^\sigma y$. Thus (ii) holds. Now suppose (ii) holds. Then $\vec{1} \sim_{\Omega \setminus E}^\sigma \vec{0}$. That is,

$$\left(\mu_i(\Omega \setminus E) + \int_E z(\omega) d\mu_i(\omega) \right)_{i=0}^{n-1} = \left(0 + \int_E z(\omega) d\mu_i(\omega) \right)_{i=0}^{n-1}$$

or $\mu_i(\Omega \setminus E) = 0$ for all i , as required. Q.E.D.

DEFINITION A.3: Say a set E is *assumed* under \succsim if E is Borel and satisfies the following conditions:

- (i) *Nontriviality*: E is nonempty and, for each open set U with $E \cap U \neq \emptyset$, there are acts $x, y \in \mathcal{A}$ with $x \succ_{E \cap U} y$;
- (ii) *Strict Determination*: For all acts $x, y \in \mathcal{A}$, $x \succ_E y$ implies $x \succ y$.

PROPOSITION A.2: Fix a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(\Omega)$ and a Borel set E in Ω . Then E is assumed under σ if and only if E is assumed under \succsim^σ .

PROOF: First suppose that E is assumed under σ at level j . Fix an open set U with $E \cap U \neq \emptyset$. Then, by conditions (ii) and (iii) of assumption, there exists some $k \leq j$ with $\mu_k(E \cap U) > 0$. Let $x(\omega) = 1$ if $\omega \in E \cap U$ and let $x(\omega) = 0$ otherwise. Then the act $(x_{E \cap U}, \vec{0}_{\Omega \setminus (E \cap U)})$ is evaluated as $(\mu_0(E \cap U), \dots, \mu_j(E \cap U), 0, \dots, 0)$, where the k th entry is strictly positive. The act $(\vec{0}_{E \cap U}, \vec{0}_{\Omega \setminus (E \cap U)})$ is evaluated as $(0, \dots, 0)$. Thus $\vec{x} \succ_{E \cap U}^\sigma \vec{0}$, establishing Nontriviality. To establish Strict Determination, note that $x \succ_E^\sigma y$ implies

$$\left(\int_E x d\mu_0, \dots, \int_E x d\mu_j, \int_{\Omega \setminus E} z d\mu_{j+1}, \dots, \int_{\Omega \setminus E} z d\mu_{n-1} \right) >^L \left(\int_E y d\mu_0, \dots, \int_E y d\mu_j, \int_{\Omega \setminus E} z d\mu_{j+1}, \dots, \int_{\Omega \setminus E} z d\mu_{n-1} \right),$$

so that certainly

$$\left(\int_E x d\mu_0, \dots, \int_E x d\mu_j, \int_{\Omega \setminus E} x d\mu_{j+1}, \dots, \int_{\Omega \setminus E} x d\mu_{n-1} \right) >^L \left(\int_E y d\mu_0, \dots, \int_E y d\mu_j(\omega), \int_{\Omega \setminus E} y d\mu_{j+1}, \dots, \int_{\Omega \setminus E} y d\mu_{n-1} \right).$$

Thus $x >^\sigma y$, establishing Strict Determination.

Next, suppose E is assumed under \succsim^σ . We want to show that E is assumed under σ . Condition (iii) of assumption is immediate from Nontriviality, so we will show that σ satisfies conditions (i) and (ii).

Assume σ fails conditions (i) and (ii) of assumption. There are three cases to consider.

CASE A.1— $\mu_i(E) = 0$ for all i : This contradicts Nontriviality.

CASE A.2— $\mu_i(E) = 0$ and $\mu_h(E) = 1$ where $h > i$: Let U_i and U_h be Borel sets as in Definition 4.1 (i.e., with $\mu_i(U_i) = 1$ and, for $i \neq k$, $\mu_i(U_k) = 0$, and similarly for h). Define

$$x(\omega) = \begin{cases} 1, & \text{if } \omega \in E \cap U_h, \\ 0, & \text{otherwise,} \end{cases}$$

$$y(\omega) = \begin{cases} 1, & \text{if } \omega \in U_i \setminus E, \\ 0, & \text{otherwise.} \end{cases}$$

Acts x and $(x_E, \vec{0}_{\Omega \setminus E})$ are evaluated as $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 corresponds to μ_h . (Here, we use $\mu_k(U_h) = 0$ for all $k \neq h$.) Act y is evaluated as $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 corresponds to μ_i , while act $(y_E, \vec{0}_{\Omega \setminus E})$ is evaluated as $(0, \dots, 0)$. Thus $x >_E^\sigma y$. But since $h > i$, $y >^\sigma x$, contradicting Strict Determination.

CASE A.3— $0 < \mu_i(E) < 1$ for some i : Let U_i be a Borel set as in Definition 4.1 and define

$$x(\omega) = \begin{cases} \mu_i(U_i \setminus E), & \text{if } \omega \in E \cap U_i, \\ 0, & \text{otherwise,} \end{cases}$$

$$y(\omega) = \begin{cases} 1, & \text{if } \omega \in U_i \setminus E, \\ 0, & \text{otherwise.} \end{cases}$$

Acts x and $(x_E, \vec{0}_{\Omega \setminus E})$ are evaluated as

$$(0, \dots, 0, \mu_i(U_i \setminus E)\mu_i(E \cap U_i), 0, \dots, 0),$$

where the nonzero entry corresponds to μ_i . This entry is indeed nonzero, since $1 > \mu_i(E) > 0$ implies $\mu_i(U_i \setminus E) > 0$ and $\mu_i(E \cap U_i) > 0$. Act y is evaluated as

$$(0, \dots, 0, \mu_i(U_i \setminus E), 0, \dots, 0),$$

where the nonzero entry corresponds to μ_i . This entry is indeed nonzero, since $1 > \mu_i(E)$. The act $(y_E, \vec{0}_{\Omega \setminus E})$ is evaluated as $(0, \dots, 0)$. Thus $x \succ_E^\sigma y$. But since $1 > \mu_i(E \cap U_i)$, $y \succ^\sigma x$, contradicting Strict Determination.

Q.E.D.

COROLLARY A.1: *Fix a full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}^+(\Omega)$ and a Borel set E in Ω . If E is believed under \succsim^σ , then E satisfies Nontriviality and Strict Determination.*

We conclude by mentioning the relationship between this axiomatization and BBD's (1991a) axiomatization. Fix a finite state space and suppose \succsim is represented by a full-support LPS. Impose BBD's Axiom 5' (i.e., their full-support condition). BBD then says that E is infinitely more likely than not- E if E is nonempty and, for all acts x, y, w, z , $x \succ_E y$ implies $(x_E, w_{\Omega \setminus E}) \succ (y_E, z_{\Omega \setminus E})$. (See their Definition 5.1.) It is easily checked that E is infinitely more likely than $\Omega \setminus E$, in the sense of BBD, if and only if Nontriviality and Strict Determination hold.

In BBD, Axiom 5' is needed to ensure that their Definition 5.1 carries the intended interpretation. (Without it, there might be no x, y with $x \succ_E y$, i.e., each measure in the LPS could assign zero probability to E .) Nontriviality plays an analogous role in our formulation.

Suppose \succsim is represented by a full-support LPS σ . Fix an event E . In the context of a finite state space, Corollary 5.1 in BBD shows that \succsim satisfies Nontriviality and Strict Determination if and only if σ satisfies conditions (i) and (ii) of assumption. For a finite state space and a full-support LPS, an event satisfies conditions (i) and (ii) of assumption if and only if it satisfies conditions (i)–(iii) of assumption. Proposition A.2 extends this result to infinite spaces.

APPENDIX B: PROOFS FOR SECTIONS 5 AND 6

This appendix provides proofs that relate to the definition and properties of assumption.

PROOF OF PROPOSITION 5.1: Suppose E is assumed under σ at level j . Condition (a) of Definition 5.1 follows immediately from condition (iii) of assumption. Next, suppose F is part of E and G is part of $\Omega \setminus E$. Suppose further that $\mu_i(F) > 0$ and $\mu_k(G) > 0$. Then, by conditions (i) and (ii) of assumption, $i \leq j < k$ as required.

For the converse, suppose conditions (a) and (b) of Definition 5.1 hold. By condition (b), whenever $\mu_i(E) > 0$ and $\mu_k(\Omega \setminus E) > 0$, we have that $i <$

k. Moreover, by condition (a), there is some i with $\mu_i(E) > 0$. This establishes that there is some j satisfying conditions (i) and (ii) of assumption. Condition (iii) of assumption is immediate from condition (a) of Definition 5.1. *Q.E.D.*

It will be useful to have the following characterization of assumption.

LEMMA B.1: *Fix a full-support LPS $\sigma \in \mathcal{L}^+(\Omega)$ and an event E . Then E is assumed under $\sigma = (\mu_0, \dots, \mu_{n-1})$ at level j if and only if there is some j so that σ satisfies conditions (i) and (ii) plus the following condition:*

(iii') $E \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$.

PROOF: First suppose that E is assumed under σ at level j . We will show that σ also satisfies (iii'). Consider the open set

$$U = \Omega \setminus \bigcup_{i \leq j} \text{Supp } \mu_i.$$

If $U \cap E \neq \emptyset$, then $\mu_i(U \cap E) > 0$ for some i . By condition (ii) of assumption, $i \leq j$. This implies that, for some $i \leq j$, $\mu_i(U) > 0$ and so $U \cap \text{Supp } \mu_i \neq \emptyset$, a contradiction. This says $U \cap E = \emptyset$ and so $E \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$, as required.

Next suppose that there is some j so that σ satisfies conditions (i) and (ii), and also (iii'). We will show that it satisfies condition (iii). Let U be an open set with $U \cap E \neq \emptyset$. By condition (iii'), for each $\omega \in U \cap E$, there is some $i \leq j$ with $\omega \in \text{Supp } \mu_i$. Since U is an open neighborhood of ω , $\mu_i(U) > 0$. By condition (i) of assumption, $\mu_i(E \cap U) = \mu_i(U) > 0$, as required. *Q.E.D.*

We now turn to establish properties of the assumption operator.

PROOF OF PROPERTY 6.1—Convexity: Let $\sigma = (\mu_0, \dots, \mu_{n-1})$, and fix events E and F that are assumed under σ at level j . Fix also a Borel set G with $E \cap F \subseteq G \subseteq E \cup F$. We will show that G is also assumed under σ at level j .

First fix $i \leq j$ and note that $\mu_i(E) = \mu_i(F) = 1$. So certainly $\mu_i(E \cap F) = 1$. Since $E \cap F \subseteq G$, $\mu_i(G) = 1$, establishing property (i) of assumption. Next fix $i > j$. Note that $\mu_i(E) = \mu_i(F) = 0$ and so $\mu_i(E \cup F) = 0$. Since $G \subseteq E \cup F$, $\mu_i(G) = 0$, establishing property (ii) of assumption. Finally, since E and F are assumed under σ at level j , Lemma B.1 says $E \cup F \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$. So using the fact that $G \subseteq E \cup F$ and Lemma B.1, G is assumed under σ . *Q.E.D.*

PROOF OF PROPERTY 6.2—Closure: Let $\sigma = (\mu_0, \dots, \mu_{n-1})$ and suppose E is assumed under σ at level j . Then $\bar{E} = \bigcup_{i \leq j} \text{Supp } \mu_i$. To see this, note that Lemma B.1 says that $E \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$. Since $\bigcup_{i \leq j} \text{Supp } \mu_i$ is closed, $\bar{E} \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$. Moreover, for all $i \leq j$, $\mu_i(\bar{E}) = 1$ so that $\bigcup_{i \leq j} \text{Supp } \mu_i \subseteq \bar{E}$.

If F is also assumed under σ at level j , then it is immediate that $\bar{E} = \bar{F}$. If F is assumed under σ at level $k > j$, then $\bar{E} \subseteq \bar{F}$, since $\bigcup_{i \leq j} \text{Supp } \mu_i \subseteq \bigcup_{i \leq k} \text{Supp } \mu_i$. Q.E.D.

PROOF OF PROPERTY 6.3—Conjunction and Disjunction: We will only prove the Conjunction property. The proof of the Disjunction property is similar.

Let $\sigma = (\mu_0, \dots, \mu_{n-1})$. For each m , E_m is assumed under σ at some level j_m . Let $j_M = \min\{j_m : m = 1, 2, \dots\}$. Then, for each m , $\mu_i(E_m) = 1$ for all $i \leq j_m$. Thus $\mu_i(\bigcap_m E_m) = 1$ for all $i \leq j_M$. Also, $\mu_i(E_M) = 0$ for all $i > j_M$. Then certainly $\mu_i(\bigcap_m E_m) = 0$ for all $i > j_M$. This establishes conditions (i) and (ii) of Proposition 5.1 (for $j = j_M$). Finally, using the fact that E_M is assumed at level j_M and Lemma B.1,

$$\bigcap_m E_m \subseteq E_M \subseteq \bigcup_{i \leq j_M} \text{Supp } \mu_i.$$

Again using Lemma B.1, this establishes condition (iii) of Proposition 5.1. Q.E.D.

APPENDIX C: PROOFS FOR SECTION 7

In what follows, we will need to make use of the following characterizations of full support.

LEMMA C.1: *A sequence $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{N}(\Omega)$ has full support if and only if, for each nonempty open set U , there is an i with $\mu_i(U) > 0$.*

PROOF: Fix a sequence $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{N}(\Omega)$ which does not have full support. Then $U = \Omega \setminus \bigcup_{i < n} \text{Supp } \mu_i$ is nonempty. The set U is open and $\mu_i(U) = 0$ for all i . For the converse, fix a full-support sequence $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{N}(\Omega)$ and a nonempty open set U . Since σ has full support, $U \cap \text{Supp } \mu_i \neq \emptyset$ for some i . Then $(\Omega \setminus U) \cap \text{Supp } \mu_i$ is closed and strictly contained in $\text{Supp } \mu_i$, so that $\mu_i((\Omega \setminus U) \cap \text{Supp } \mu_i) < 1$. From this, $\mu_i(U) > 0$, as required. Q.E.D.

In the next three lemmas, Borel without qualification means Borel in $\mathcal{N}(\Omega)$. We make repeated use of the following facts:

- (i) There is a countable open basis E_1, E_2, \dots for Ω .
- (ii) For each Borel set B in Ω and $r \in [0, 1]$, the set of μ such that $\mu(B) > r$ is Borel in $\mathcal{M}(\Omega)$.
- (iii) For each Borel set Y in $\mathcal{M}(\Omega)$ and each k , the set of $\sigma = (\mu_0, \dots, \mu_{n-1})$ in $\mathcal{N}(\Omega)$ such that $n > k$ and $\mu_k \in Y$ is Borel.

Fact (i) follows from the assumption that Ω is separable. Fact (ii) says that the function $\mu \mapsto \mu(B)$ is Borel, which follows from Kechris (1995, Theorem 17.24). Fact (iii) follows from the continuity of the projection function $\sigma \mapsto \mu_k$ from $\mathcal{N}(\Omega)$ to $\mathcal{M}(\Omega)$.

Let $\mathcal{N}_n(\Omega)$ be the set of all σ in $\mathcal{N}(\Omega)$ of length n , and define $\mathcal{N}_n^+(\Omega)$, $\mathcal{L}_n(\Omega)$, and $\mathcal{L}_n^+(\Omega)$ analogously.

LEMMA C.2: *Fix $n \in \mathbb{N}$. For any Polish space Ω , the sets $\mathcal{N}_n(\Omega)$, $\mathcal{N}_n^+(\Omega)$, $\mathcal{L}_n(\Omega)$, and $\mathcal{L}_n^+(\Omega)$ are Borel.*

PROOF: In this proof, $\sigma = (\mu_0, \dots, \mu_{n-1})$ varies over $\mathcal{N}_n(\Omega)$. Recall that if $\sigma \in \mathcal{N}_n(\Omega)$ and $\tau \in \mathcal{N}(\Omega) \setminus \mathcal{N}_n(\Omega)$, then σ has distance 1 from τ . Thus $\mathcal{N}_n(\Omega)$ is open and hence Borel.

By Lemma C.1 and fact (i), a sequence $\sigma \in \mathcal{N}_n(\Omega)$ has full support if and only if, for each basic open set E_i , there exists $j < n$ such that $\mu_j(E_i) > 0$. By facts (ii) and (iii), for each i and j the set of σ such that $\mu_j(E_i) > 0$ is Borel. Therefore, $\mathcal{N}_n^+(\Omega)$ is Borel.

Write $\mu \perp \nu$ if there is a Borel set $U \subseteq \Omega$ such that $\mu(U) = 1$ and $\nu(U) = 0$. It is easy to see that mutual singularity holds for an element $\sigma \in \mathcal{N}_n(\Omega)$ if and only if $\mu_i \perp \mu_j$ for all $i < j$. To prove that $\mathcal{L}_n(\Omega)$ is Borel, it suffices to prove that for each $i < j$, the set of σ such that $\mu_i \perp \mu_j$ is Borel. Note that $\mu_i \perp \mu_j$ if and only if for each m , there is an open set V such that $\mu_i(V) = 1$ and $\mu_j(V) < \frac{1}{m}$. By fact (i), this in turn holds if and only if for each m there exists k such that $\mu_i(E_k) > 1 - \frac{1}{m}$ and $\mu_j(E_k) < \frac{1}{m}$. By facts (ii) and (iii), the set of σ such that $\mu_i(E_k) > 1 - \frac{1}{m}$ is Borel, and the set of σ such that $\mu_j(E_k) < \frac{1}{m}$ is Borel. The set of σ such that $\mu_i \perp \mu_j$ is a Borel combination of these sets, and hence is Borel, as required. Thus $\mathcal{L}_n(\Omega)$ is Borel.

Since $\mathcal{L}_n^+(\Omega)$ is the intersection of the Borel sets $\mathcal{N}_n^+(\Omega)$ and $\mathcal{L}_n(\Omega)$, it is also Borel. Q.E.D.

COROLLARY C.1: *For any Polish space Ω , the sets $\mathcal{N}^+(\Omega)$, $\mathcal{L}(\Omega)$, and $\mathcal{L}^+(\Omega)$ are Borel.*

PROOF: Each $\mathcal{N}_n^+(\Omega)$ is Borel and $\mathcal{N}^+(\Omega) = \bigcup_n \mathcal{N}_n^+(\Omega)$. Likewise for $\mathcal{L}(\Omega)$ and $\mathcal{L}^+(\Omega)$. Q.E.D.

LEMMA C.3: *For each Polish space Ω and Borel set E in Ω , the set of $\sigma \in \mathcal{L}^+(\Omega)$ such that E is assumed under σ is Borel.*

PROOF: Fix n and $j < n$. By fact (ii), the sets of μ such that $\mu(E) = 1$ and such that $\mu(E) = 0$ are Borel in $\mathcal{M}(\Omega)$. Therefore, by fact (iii) and Corollary C.1, the set of $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}_n^+(\Omega)$ such that conditions (i) and (ii) in Proposition 5.1 hold is Borel. Let $\{d_0, d_1, \dots\}$ be a countable dense subset of E . For each k and $\mu \in \mathcal{M}(\Omega)$, we have $d_k \in \text{Supp } \mu$ if and only if $\mu(B) > 0$ for

every open ball B with center d_k and rational radius. Then by fact (ii), the set of μ such that $d_k \in \text{Supp } \mu$ is Borel in $\mathcal{M}(\Omega)$. We have $E \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$ if and only if $d_k \in \bigcup_{i \leq j} \text{Supp } \mu_i$ for all $k \in \mathbb{N}$. Therefore, the set of $\sigma \in \mathcal{L}_n^+(\Omega)$ with $E \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$ is Borel. By Lemma B.1, the set of $\sigma \in \mathcal{L}^+(\Omega)$ such that E is assumed under σ is Borel. Q.E.D.

LEMMA C.4: For each m :

- (i) $R_m^a = R_1^a \cap [S^a \times \bigcap_{i < m} A^a(R_i^b)]$.
- (ii) R_m^a is Borel in $S^a \times T^a$.

PROOF: Part (i) is immediate.

Part (ii) is by induction. For $m = 1$, first note that since λ^a is Borel measurable, Lemma C.2 says that for each n the set $(\lambda^a)^{-1}(\mathcal{L}_n^+(S^b \times T^b))$ is Borel in T^a . From Definition 7.4, for each $s^a \in S^a$ there is a finite Boolean combination C of linear equations in $n \cdot |S^b|$ variables such that whenever $\lambda^a(t^a) = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{L}_n^+(S^b \times T^b)$, the pair (s^a, t^a) is rational if and only if C holds for $\{\text{marg}_{S^b} \mu_i(s^b) : i < n, s^b \in S^b\}$. Since S^a and S^b are finite, this shows that R_1^a is Borel in $S^a \times T^a$.

Assume the result holds for all $i \leq m$. Then, by Lemma C.3, for each $i \leq m$, $A^a(R_i^b)$ is Borel in T^a . So R_{m+1}^a is Borel. Q.E.D.

PROOF OF PROPOSITION 7.1: (i) Start with a type structure $\langle S^a, S^b, T^a, T^b, \kappa^a, \kappa^b \rangle$. The case that $S^b \times T^b$ is a singleton is trivial, so we may assume that it is not. Pick any $\sigma \in \mathcal{L}(S^b \times T^b)$ which does not have full support. Define $\lambda^a(t^a) = \kappa^a(t^a)$ if $\kappa^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$ and $\lambda^a(t^a) = \sigma$ otherwise. Since $\mathcal{L}^+(S^b \times T^b)$ is Borel, λ^a is a Borel map. Define λ^b similarly.

(ii) It is clear from the definitions that the two structures have the same rationality sets R_1^a and R_1^b . By induction, they also have the same sets R_m^a and R_m^b : Types associated with the R_m^a and R_m^b sets are all associated with full-support LPS's, so that only assumption by full-support LPS's is involved. Q.E.D.

PROOF OF PROPOSITION 7.2: Let T^a and T^b be the Baire space—that is, the metric space $\mathbb{N}^{\mathbb{N}}$ with the product metric, where \mathbb{N} has the discrete metric. There is a continuous surjection λ^a (resp. λ^b) from T^a (resp. T^b) onto any Polish space, in particular onto $\overline{\mathcal{L}}(S^b \times T^b)$ (resp. $\overline{\mathcal{L}}(S^b \times T^b)$). (See Kechris (1995, p. 13 and Theorem 7.9).) These maps give us a complete type structure. Q.E.D.

APPENDIX D: PROOFS FOR SECTION 8

LEMMA D.1: Suppose t^a assumes $E \subseteq S^b \times T^b$ at level j , where $\lambda^a(t^a) = (\mu_0, \dots, \mu_{n-1})$. Then $\bigcup_{i \leq j} \text{Supp marg}_{S^b} \mu_i = \text{proj}_{S^b} E$.

PROOF: Fix $s^b \in \text{proj}_{S^b} E$, that is, $(s^b, t^b) \in E$ for some t^b . Then $\{s^b\} \times T^b$ is an open neighborhood of (s^b, t^b) . So, by conditions (ii) and (iii) of Proposition 5.1, there is some $i \leq j$ with $\mu_i(E \cap (\{s^b\} \times T^b)) > 0$. Therefore, $0 < \mu_i(\{s^b\} \times T^b) = \text{marg}_{S^b} \mu_i(s^b)$ and hence $s^b \in \text{Supp marg}_{S^b} \mu_i$. Next fix $s^b \notin \text{proj}_{S^b} E$. Then $\{s^b\} \times T^b$ is disjoint from E . But for each $i \leq j$, we have $\mu_i(E) = 1$, so $\mu_i(\{s^b\} \times T^b) = \text{marg}_{S^b} \mu_i(s^b) = 0$ and hence $s^b \notin \text{Supp marg}_{S^b} \mu_i$. Q.E.D.

The next series of lemmas concerns the geometry of polytopes. We will first review some notions from geometry, then state the lemmas, then explain the connection between the geometric notions and games, then present some intuitive examples, and finally give the formal proofs of the lemmas.

Throughout this section, we will fix a finite set $X = \{x_1, \dots, x_n\} \subseteq \mathbb{R}^d$. The *polytope* generated by X , denoted by P , is the closed convex hull of X —that is, the set of all sums $\sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \geq 0$ for each i and $\sum_{i=1}^n \lambda_i = 1$. The *affine hull* of P , denoted by $\text{aff}(P)$, is the set of all affine combinations of finitely many points in P —that is, the set of all sums $\sum_{i=1}^k \lambda_i y_i$, where $y_1, \dots, y_k \in P$ and $\sum_{i=1}^k \lambda_i = 1$. The *relative interior* of P , denoted by $\text{relint}(P)$, is the set of all $x \in \text{aff}(P)$ such that there is an open ball $B(x)$ centered around x , with $\text{aff}(P) \cap B(x) \subseteq P$.

A *hyperplane* in \mathbb{R}^d is a set of the form $H(u, \alpha) = \{x \in \mathbb{R}^d : \langle x, u \rangle = \alpha\}$ for some nonzero $u \in \mathbb{R}^d$. A hyperplane $H(u, \alpha)$ *supports* a polytope P if $\alpha = \sup\{\langle x, u \rangle : x \in P\}$. A *face* of P is either P itself or a set of the form $H \cap P$, where H is a hyperplane that supports P . If $F \neq P$ is a face of P , we say F is a *proper face*. A face $H \cap P$ is *strictly positive* if $H = H(u, \alpha)$ for some (u, α) such that each coordinate of u is strictly positive.

Given a point x in a polytope P , say the points $x_1, \dots, x_k \in P$ each *support* $x \in P$ if there are $\lambda_1, \dots, \lambda_k$, with $0 < \lambda_i \leq 1$ for each i , $\sum_{i=1}^k \lambda_i = 1$, and $x = \sum_{i=1}^k \lambda_i x_i$. Write $\text{su}(x)$ for the set of points that support $x \in P$. (Note the slight abuse of notation relative to that introduced before Definition 3.3.)

Here are the lemmas we will need:

LEMMA D.2: *If F is a face of a polytope P and $x \in F$, then $\text{su}(x) \subseteq F$.*

LEMMA D.3: *For each point x in a polytope P , $\text{su}(x)$ is a face of P .*

LEMMA D.4: *If x belongs to a strictly positive face of a polytope P , then $\text{su}(x)$ is a strictly positive face of P .*

We now give the interpretation of the geometric notions in game theory. Let d be the cardinality of the finite strategy set S^b . Each strategy $s^a \in S^a$ corresponds to the point

$$\vec{\pi}^a(s^a) = (\pi^a(s^a, s^b) : s^b \in S^b) \in \mathbb{R}^d.$$

For any probability measure $\mu \in \mathcal{M}(S^a)$, $\vec{\pi}^a(\mu)$ is the point

$$\vec{\pi}^a(\mu) = \sum_{s^a \in S^a} \mu(s^a) \vec{\pi}^a(s^a).$$

Notice that $\vec{\pi}^a(\mu)$ is in the polytope P generated by the finite set $\{\vec{\pi}^a(s^a) : s^a \in S^a\}$.

Let us identify each probability measure $\nu \in \mathcal{M}(S^b)$ with the point $(\nu(s^b) : s^b \in S^b) \in \mathbb{R}^d$. Then for each pair $(\mu, \nu) \in \mathcal{M}(S^a) \times \mathcal{M}(S^b)$, $\langle \vec{\pi}^a(\mu), \nu \rangle$ is the expected payoff to Ann. Thus, a pair (μ, ν) gives expected payoff α to Ann if and only if $\vec{\pi}^a(\mu)$ belongs to the hyperplane $H(\nu, \alpha)$. It follows that a set F is a strictly positive face of P if and only if there is a probability measure ν with support S^b such that

$$F = \{\vec{\pi}^a(\mu) : \mu \in \mathcal{M}(S^a) \text{ is optimal under } \nu\}.$$

Consider an admissible strategy s^a . By Lemma 3.1, $\vec{\pi}^a(s^a)$ is optimal under some measure ν with support S^b . That is, $\vec{\pi}^a(s^a)$ belongs to some strictly positive face of P . Lemma D.4 shows that $\text{su}(\vec{\pi}^a(s^a))$ is a strictly positive face of P . So we can pick ν so that, for every $r^a \in S^a$, $\vec{\pi}^a(r^a)$ is optimal under ν if and only if $\vec{\pi}^a(r^a) \in \text{su}(\vec{\pi}^a(s^a))$. This is the fact we use in the proof of Theorem 8.1(ii).

We next give some intuition for Lemmas D.2–D.4. Let P be a tetrahedron, as in Figure D.1. The point x^* is supported by the hyperplane H , and the corresponding face $H \cap P$ is the shaded region shown. The set of points that support x^* , that is, the set $\text{su}(x^*)$, is the line segment from x_2 to x_4 . Note that these points are also contained in the face $H \cap P$. The general counterpart of this is Lemma D.2.

Now a converse. In Figure D.1, the point x_3 lies in $H \cap P$, but does not support x^* . However, we can tilt the hyperplane H to get a new supporting hyperplane H' as in Figure D.2. Here, $H' \cap P$ is the line segment from x_2 to x_4 , that is, exactly the set $\text{su}(x^*)$. The general counterpart is Lemma D.3.

Consider another example in Figure D.3. Here P is the line segment from $(1, 0)$ to $(1, 1)$. Note that $\text{su}((1, 0)) = \{(1, 0)\}$. The hyperplane H supports $(1, 0)$, and $H \cap P = P$. We can tilt the hyperplane to get H' , where $H' \cap P = \{(1, 0)\}$ (in accordance with Lemma D.3). But note that we cannot do this if we require the hyperplane to be nonnegative. (Indeed, H is the unique nonnegative hyperplane supporting $(1, 0)$.) Intuitively, though, we will have room to tilt the hyperplane and maintain nonnegativity—in fact, strict positivity—if the original hyperplane is strictly positive. This is Lemma D.4.

We now turn to the proofs of Lemmas D.2–D.4.

PROOF OF LEMMA D.2: Fix a face F that contains x . If $F = P$, then certainly $\text{su}(x) \subseteq F$. If $F \neq P$, there is a hyperplane $H = H(u, \alpha)$ that supports P with $F = H \cap P$. Fix $y \in \text{su}(x)$. Then there are $x_1, \dots, x_k \in P$ and

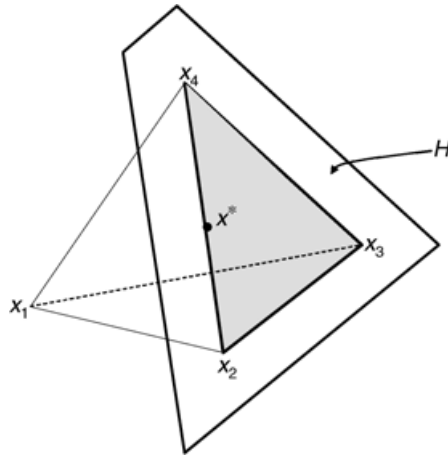


FIGURE D.1.

$\lambda_1, \dots, \lambda_k$ with $0 < \lambda_i \leq 1$ for each i , $\sum_{i=1}^k \lambda_i = 1$, $y = x_1$, and $x = \sum_{i=1}^k \lambda_i x_i$. Let $z = \sum_{i=2}^k (\lambda_i / (1 - \lambda_1)) x_i$ and note that $z \in P$, since P is convex. Also note that $x = \lambda_1 y + (1 - \lambda_1) z$; that is, x lies on the line segment from y to z .

Since $x \in H$ and $y, z \in P$,

$$\langle x, u \rangle = \alpha, \quad \langle y, u \rangle \leq \alpha, \quad \langle z, u \rangle \leq \alpha.$$

Moreover, since x lies on the line segment from y to z ,

$$\langle y, u \rangle \leq \langle x, u \rangle \leq \langle z, u \rangle.$$

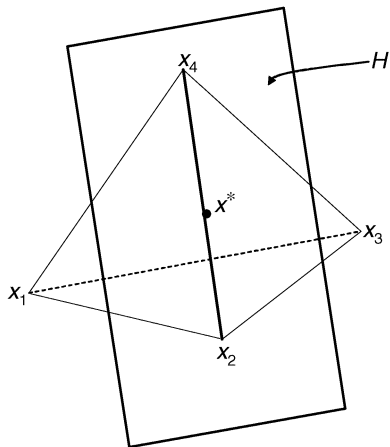


FIGURE D.2.

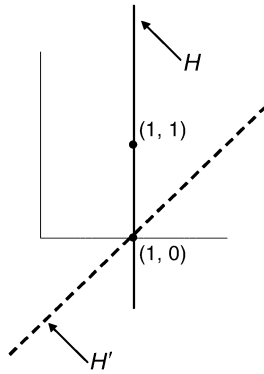


FIGURE D.3.

It follows that $\langle y, u \rangle = \alpha$, so $y \in F$. *Q.E.D.*

For the next proofs we need the following basic facts about a general polytope P (see Ziegler (1998, Chap. 2)):

- P1: Every face of P is a polytope.
- P2: Every face of a face of P is a face of P .
- P3: If $x \in P$, either $x \in \text{relint}(P)$ or x belongs to a proper face of P .
- P4: P has finitely many faces.

We record an immediate consequence of P1–P4.

LEMMA D.5: *If $x \in P$, then there exists a face F of P with $x \in \text{relint}(F)$.*

PROOF: If $x \in \text{relint}(P)$, the result holds trivially. So suppose $x \notin \text{relint}(P)$. By P3, x is contained in some proper face F of P . By P1, the face F is a polytope. Using P2 and P4, we can choose F so that there does not exist a proper face of F that contains x . P3 then implies $x \in \text{relint}(F)$. *Q.E.D.*

The next lemma establishes a fact about points in the relative interior of a face F of P .

LEMMA D.6: *Let F be a face of P . If $x \in \text{relint}(F)$, then $F \subseteq \text{su}(x)$.*

PROOF: Fix $x \in \text{relint}(F)$ and some $x' \in F$. If $x' = x$, then certainly $x' \in \text{su}(x)$. If not, consider the line going through both x and x' , to be denoted by $L(x, x')$. Since $x \in \text{relint}(F)$, there is some open ball $B(x)$ centered around

x , with $\text{aff}(F) \cap B(x) \subseteq F$. Then $\text{aff}(F) \cap B(x)$ must meet $L(x, x')$. Certainly, we can find a point x'' both on $L(x, x')$ and in $\text{aff}(F) \cap B(x)$, with $d(x', x) < d(x', x'')$ for the Euclidean metric d . Then there must exist $0 < \lambda < 1$ with $x = \lambda x' + (1 - \lambda)x''$. Since $x', x'' \in P$, this establishes $x' \in \text{su}(x)$. *Q.E.D.*

We now turn to the proofs of Lemmas D.3 and D.4.

PROOF OF LEMMA D.3: Fix $x \in P$. By Lemma D.5, there exists a face F of P with $x \in \text{relint}(F)$. We then have $\text{su}(x) \subseteq F$ by Lemma D.2 and $F \subseteq \text{su}(x)$ by Lemma D.6. *Q.E.D.*

PROOF OF LEMMA D.4: Let $H(u, \alpha) \cap P$ be a strictly positive face of P containing x . By Lemma D.3, $\text{su}(x) = H(u', \alpha') \cap P$ is a face of P . Set

$$u'' = u' + \beta u, \quad \alpha'' = \alpha' + \beta \alpha$$

for some $\beta > 0$. If $y \in H(u', \alpha') \cap P$, we get

$$\langle y, u'' \rangle = \langle y, u' \rangle + \beta \langle y, u \rangle = \alpha' + \beta \alpha = \alpha'',$$

using $\text{su}(x) \subseteq H(u, \alpha) \cap P$. If $y \in P \setminus H(u', \alpha')$, we get

$$\langle y, u'' \rangle = \langle y, u' \rangle + \beta \langle y, u \rangle < \alpha' + \beta \langle y, u \rangle \leq \alpha' + \beta \alpha = \alpha''.$$

Thus $H(u'', \alpha'')$ is a supporting hyperplane with $\text{su}(x) = H(u'', \alpha'') \cap P$. Moreover, we can choose $\beta > 0$ so that $u'' \gg 0$ as required. *Q.E.D.*

APPENDIX E: PROOFS FOR SECTION 9

LEMMA E.1: *If $s^a \in S_m^a$, then there exists $\mu \in \mathcal{M}(S^b)$ with $\text{Supp } \mu = S_{m-1}^b$, such that $\pi^a(s^a, \mu) \geq \pi^a(r^a, \mu)$ for each $r^a \in S^a$.*

PROOF: By Lemma 3.1, there exists $\mu \in \mathcal{M}(S^b)$ with $\text{Supp } \mu = S_{m-1}^b$, such that $\pi^a(s^a, \mu) \geq \pi^a(r^a, \mu)$ for all $r^a \in S_{m-1}^a$. Suppose there is an $r^a \in S^a \setminus S_{m-1}^a$ with

$$(E.1) \quad \pi^a(s^a, \mu) < \pi^a(r^a, \mu).$$

We have $r^a \in S_l^a \setminus S_{l+1}^a$ for some $l < m - 1$. Choose r^a (and l) so that there does not exist $q^a \in S_{l+1}^a$ with $\pi^a(s^a, \mu) < \pi^a(q^a, \mu)$.

Fix some $\nu \in \mathcal{M}(S^b)$ with $\text{Supp } \nu = S_l^b$ and define a sequence of measures $\mu^n \in \mathcal{M}(S^b)$, for each $n \in \mathbb{N}$, by $\mu^n = (1 - \frac{1}{n})\mu + \frac{1}{n}\nu$. Note that $\text{Supp } \mu^n = S_l^b$ for each n . Using $r^a \notin S_{l+1}^a$ and Lemma 3.1 applied to the $(l + 1)$ -admissible strategies, it follows that for each n there is a $q^a \in S_l^a$ with

$$(E.2) \quad \pi^a(q^a, \mu^n) > \pi^a(r^a, \mu^n).$$

We can assume that $q^a \in S_{l+1}^a$. (Choose $q^a \in S_l^a$ to maximize the left-hand side of equation (E.2) among all strategies in S_l^a .) Also, since S_{l+1}^a is finite, there is a $q^a \in S_{l+1}^a$ such that equation (E.2) holds for infinitely many n . Letting $n \rightarrow \infty$ yields

$$(E.3) \quad \pi^a(q^a, \mu) \geq \pi^a(r^a, \mu).$$

From (E.1) and (E.3) we get $\pi^a(q^a, \mu) > \pi^a(s^a, \mu)$, contradicting our choice of r^a . Q.E.D.

The next lemma will guarantee that we will have enough room to build the measures we need to establish Lemma E.3. For $t^a, u^a \in T^a$, write $t^a \approx u^a$ if, for each i , the component measures $(\lambda^a(t^a))_i$ and $(\lambda^a(u^a))_i$ have the same marginals on S^b , and are mutually absolutely continuous (have the same null sets).

LEMMA E.2: *In a complete type structure:*

- (i) *If $\lambda^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$ and $u^a \approx t^a$, then $\lambda^a(u^a) \in \mathcal{L}^+(S^b \times T^b)$.*
- (ii) *If $\lambda^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$, then there are continuum many u^a such that $u^a \approx t^a$.*
- (iii) *For each set $E \subseteq S^b \times T^b$, the set $A^a(E)$ is closed under the relation \approx . In fact, for each j , if $t^a \approx u^a$ and E is assumed under $\lambda^a(t^a)$ at level j , then E is assumed under $\lambda^a(u^a)$ at level j .*
- (iv) *If $t^a \approx u^a$, then for each m and $s^a \in S^a$, $(s^a, t^a) \in R_m^a$ if and only if $(s^a, u^a) \in R_m^a$.*

PROOF: Part (i) follows from the fact that $\lambda^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$ and the mutual absolute continuity of the component measures of $\lambda^a(t^a)$ and $\lambda^a(u^a)$. For part (ii), note that full support implies that $\mu_i = (\lambda^a(t^a))_i$ has infinite support for some i . Therefore, there are continuum many different measures ν_i with the same null sets and marginal on S^b as μ_i . The sequence of measures obtained by replacing μ_i by ν_i belongs to $\mathcal{L}^+(S^b \times T^b)$, and by completeness this sequence is equal to $\lambda^a(u^a)$ for some u^a . It follows that, for each such u^a , $u^a \approx t^a$. For part (iii), fix $\lambda^a(t^a)$ that assumes E at level j . It follows immediately from part (i) and the mutual absolute continuity of the component measures that if $u^a \approx t^a$, then $\lambda^a(u^a)$ also assumes E at level j . For part (iv), the case of $m = 1$ follows immediately from part (i). The case of $m > 1$ is proved by induction and makes use of part (iii). Q.E.D.

$$\text{Set } R_0^a = S^a \times T^a \text{ and } R_0^b = S^b \times T^b.$$

LEMMA E.3: *In a complete type structure, $\text{proj}_{S^a} R_m^a = \text{proj}_{S^a} (R_m^a \setminus R_{m+1}^a)$ for each $m \geq 0$.*

PROOF: The proof is by induction on m .

$m = 0$: Choose t^a so that $\lambda^a(t^a) \notin \mathcal{L}^+(S^b \times T^b)$ and note that $S^a \times \{t^a\}$ is disjoint from R_1^a . So, $\text{proj}_{S^a}(R_0^a \setminus R_1^a) = S^a$.

$m = 1$: Fix $(s^a, t^a) \in R_1^a$. It suffices to show that there is a type $u^a \in T^a$ with $(s^a, u^a) \in R_1^a \setminus R_2^a$. To see this, first notice that there is a full-support LPS (μ) of length 1 such that s^a is optimal under (μ) . (This is by Lemma 7.1.) By completeness, there is a type u^b such that $\lambda^b(u^b) \notin \mathcal{L}^+(S^a \times T^a)$. Construct a probability measure $\nu \in \mathcal{M}(S^b \times T^b)$ with $\text{marg}_{S^b} \mu = \text{marg}_{S^b} \nu$ and $\nu(S^b \times \{u^b\}) = 1$. Let ρ be the measure $(\mu + \nu)/2$. Then ρ is a full-support LPS, so by completeness there is a type $u^a \in T^a$ with $\lambda^a(u^a) = (\rho)$. Note that s^a is optimal under (ρ) , so $(s^a, u^a) \in R_1^a$. But $\rho(R_1^b) \leq \frac{1}{2}$ because $\lambda^b(u^b) \notin \mathcal{L}^+(S^a \times T^a)$. So R_1^b is not assumed under (ρ) and therefore $(s^a, u^a) \notin R_2^a$.

$m \geq 2$: Assume the result holds for $m - 1$. Let $(s^a, t^a) \in R_m^a$ and $\lambda^a(t^a) = \sigma = (\mu_0, \dots, \mu_{n-1})$. Then $t^a \in A^a(R_i^b)$ for each $i < m$. We will find a type u^a such that $(s^a, u^a) \in R_m^a \setminus R_{m+1}^a$.

By the induction hypothesis and the fact that S^b is finite, there is a finite set $U \subseteq R_{m-1}^b \setminus R_m^b$ with $\text{proj}_{S^b} U = \text{proj}_{S^b} R_{m-1}^b$. Since $m \geq 2$, $U \subseteq R_1^b$, so $\lambda^b(t^b) \in \mathcal{L}^+(S^a \times T^a)$ for each $(s^b, t^b) \in U$. By Lemma E.2(ii), for each $(s^b, t^b) \in U$ there are continuum many u^b such that $u^b \approx t^b$, and hence there is a $u^b \approx t^b$ such that $\mu_i(\{s^b, u^b\}) = 0$ for all i . Form U' by replacing each $(s^b, t^b) \in U$ by a pair (s^b, u^b) with $u^b \approx t^b$ and $\mu_i(\{s^b, u^b\}) = 0$ for all i . Then U' is finite with $\mu_i(U') = 0$ for all i . By Lemma E.2(iv), $U' \subseteq R_{m-1}^b \setminus R_m^b$ and $\text{proj}_{S^b} U' = \text{proj}_{S^b} R_{m-1}^b$. It follows that the set U can be chosen so that $\mu_i(U) = 0$ for all i .

We will get a point $(s^a, u^a) \in R_m^a \setminus R_{m+1}^a$ by adding a measure to the beginning of the sequence σ . Since U is finite, $\text{proj}_{S^b} U = \text{proj}_{S^b} R_{m-1}^b$, and $\mu_0(R_{m-1}^b) = 1$, there is a probability measure ν such that $\nu(U) = 1$ and $\text{marg}_{S^b} \nu = \text{marg}_{S^b} \mu_0$. Let τ be the sequence $(\nu, \mu_0, \dots, \mu_{n-1})$. Since $\sigma \in \mathcal{L}^+(S^b \times T^b)$ and $\mu_i(U) = 0$ for each i , we see that $\tau \in \mathcal{L}^+(S^b \times T^b)$. By completeness there is a $u^a \in T^a$ with $\lambda^a(u^a) = \tau$. Since ν has the same marginal on S^b as μ_0 and since $(s^a, t^a) \in R_1^a$, we have $(s^a, u^a) \in R_1^a$. Since $U \subseteq R_{m-1}^b$ and $t^a \in A^a(R_k^b)$ for each $k < m$, it follows that $u^a \in A^a(R_k^b)$ for each $k < m$. Then by Lemma C.4(i), we have $(s^a, u^a) \in R_m^a$. However, since U is disjoint from R_m^b we have $\nu(R_m^b) = 0$, so $u^a \notin A^a(R_m^b)$ and hence $(s^a, u^a) \notin R_{m+1}^a$. This completes the induction. *Q.E.D.*

APPENDIX F: PROOFS FOR SECTION 10

For the following two lemmas we assume that $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ is a complete type structure in which the maps λ^a and λ^b are continuous.

LEMMA F.1: *If player a is not indifferent, then $R_0^a \setminus \overline{R_1^a}$ is uncountable.*

PROOF: We have that $\pi^a(r^a, s^b) < \pi^a(s^a, s^b)$ for some r^a, s^a, s^b . Then S^a has more than one element and by completeness, T^b has more than one element. Therefore, using completeness again, there is a type $t^a \in T^a$ such that $\lambda^a(t^a) = (\mu_0, \mu_1)$ is a full-support LPS of length 2 and $\mu_0(\{s^b\} \times T^b) = 1$. Let U be the

set of all $u^a \in T^a$ such that r^a is not optimal under $(\lambda^a(u^a))_0$, i.e., for some $q^a \in S^a$,

$$\begin{aligned} & \sum_{s^b \in S^b} \pi^a(r^a, s^b) \text{marg}_{S^b}(\lambda^a(u^a))_0(s^b) \\ & < \sum_{s^b \in S^b} \pi^a(q^a, s^b) \text{marg}_{S^b}(\lambda^a(u^a))_0(s^b). \end{aligned}$$

We now show that $t^a \in U$. Note first that since $\mu_0(\{s^b\} \times T^b) = 1$, the function $\text{marg}_{S^b}(\lambda^a(t^a))_0$ has value 1 at s^b and 0 everywhere else in S^b . Therefore, for each $q^a \in S^a$,

$$\sum_{s^b \in S^b} \pi^a(q^a, s^b) \text{marg}_{S^b}(\lambda^a(t^a))_0(s^b) = \pi^a(q^a, s^b).$$

Since $\pi^a(r^a, s^b) < \pi^a(s^a, s^b)$, the inequality defining U holds with $(q^a, u^a) = (s^a, t^a)$ and hence $t^a \in U$.

We next show that U is open. Since λ^a is continuous, the function $u^a \mapsto (\lambda^a(u^a))_0$ is continuous. Convergence in the Prohorov metric is equivalent to weak convergence, so the function

$$u^a \mapsto \text{marg}_{S^b}(\lambda^a(u^a))_0(s^b) = \int \mathbf{1}(\{s^b\} \times T^b) d(\lambda^a(u^a))_0$$

is continuous. Thus U is defined by a strict inequality between two continuous real functions of u^a and hence U is open.

Since $\{r^a\}$ is open in S^a , the set $\{r^a\} \times U$ is open in $S^a \times T^a$. By definition, the set $\{r^a\} \times U$ is disjoint from R_1^a . Now suppose $u^a \approx t^a$. Then $(\lambda^a(u^a))_0$ has the same marginals as $(\lambda^a(t^a))_0$, so $u^a \in U$ and hence $(r^a, u^a) \in \{r^a\} \times U$. Since $\{r^a\} \times U$ is open and disjoint from R_1^a , we have $(r^a, u^a) \notin \overline{R_1^a}$. By Lemma E.2, there are uncountably many u^a such that $u^a \approx t^a$, so $R_0^a \setminus \overline{R_1^a}$ is uncountable. Q.E.D.

LEMMA F.2: *Suppose that $m \geq 1$ and $R_{m-1}^b \setminus \overline{R_m^b}$ is uncountable. Then $R_m^a \setminus \overline{R_{m+1}^a}$ is uncountable.*

PROOF: The proof is similar to the proof of Lemma E.3. Fix $(s^a, t^a) \in R_m^a$. By the proof of Theorem 9.1, we can choose t^a so that $\lambda^a(t^a) = \sigma = (\mu_0, \dots, \mu_{m-1})$ and R_{m-1}^b is assumed at level 0. We will get uncountably many points $(s^a, u^a) \in R_m^a \setminus \overline{R_{m+1}^a}$ by adding one more measure to the beginning of the sequence σ and using Lemma E.2.

We claim that there is a finite set $U \subseteq R_{m-1}^b \setminus \overline{R_m^b}$ such that $\text{proj}_{S^b} U = \text{proj}_{S^b} R_{m-1}^b$ and $\mu_i(U) = 0$ for all $i < m$.

$m = 1$: Recall that, for each $(s^a, t^a) \in R_1^a$, there is a u^a such that $\lambda^a(u^a)$ is a full-support LPS and $(s^a, u^a) \in R_1^a \setminus R_2^a$. (This was shown in the proof of Lemma E.3.) The claim for $m = 1$ now follows from Lemma E.2 and the fact that S^a is finite.

$m \geq 2$: The claim was already established in the induction step of Lemma E.3.

Now, since $R_{m-1}^b \setminus \overline{R_m^b}$ is uncountable, there is a point $(s^b, t^b) \in R_{m-1}^b \setminus \overline{R_m^b}$ such that $\mu_i(s^b, t^b) = 0$ for all $i < m$. Therefore, we may also take U to contain such a point (s^b, t^b) . Let ν be a probability measure such that $\nu(U) = 1$, $\text{marg}_{S^b} \nu = \text{marg}_{S^b} \mu_0$, and $\nu(s^b, t^b) = \text{marg}_{S^b} \mu_0(s^b)$. Since R_{m-1}^b is assumed under σ at level 0, we have $(s^b, t^b) \in \text{Supp } \mu_0$, and thus $\mu_0(\{s^b\} \times T^b) = \text{marg}_{S^b} \mu_0(s^b) > 0$. Therefore, $\nu(s^b, t^b) > 0$.

Let τ be the sequence $(\nu, \mu_0, \dots, \mu_{m-1})$. Since $(s^a, t^a) \in R_1^a$, $\lambda^a(t^a) = (\mu_0, \dots, \mu_{m-1})$ is a full-support LPS. Also, $\mu_i(U) = 0$ for each i . Therefore, τ is mutually singular and so a full-support LPS. By completeness, there is a $v^a \in T^a$ with $\lambda^a(v^a) = \tau$. Then $(\lambda^a(v^a))_0 = \nu$. As in Lemma E.3, we have $(s^a, v^a) \in R_m^a$. Given this, the proof of Lemma E.2(ii) shows that there are uncountably many $u^a \approx v^a$ such that $(\lambda^a(u^a))_0 = \nu$.

Suppose $u^a \approx v^a$ and $(\lambda^a(u^a))_0 = \nu$. Then $\lambda^a(u^a)$ has length $m + 1$. By Lemma E.2, we have $(s^a, u^a) \in R_m^a$. However, since $(s^b, t^b) \notin \overline{R_m^b}$, the measure ν has an open neighborhood W , where, for each $\nu' \in W$, $\nu'(R_m^b) < 1$. (An example of such a neighborhood is the set $\{\nu' : \nu'(V) > \nu(s^b, t^b)/2\}$, where V is an open neighborhood of (s^b, t^b) which is disjoint from R_m^b .) Then the set

$$X = \{\xi \in \mathcal{N}_{m+1}(S^b \times T^b) : \xi_0 \in W\}$$

is an open neighborhood of $\lambda^a(u^a)$, and no LPS $\xi \in X$ can assume R_m^b at level 0. It follows that an LPS $\xi \in X$ cannot assume all of the $m + 1$ sets R_k^b , $k \leq m$, because by the inductive hypothesis all these sets have different closures, and hence by Property 6.2 at most one can be assumed at each level. By continuity of λ^a , the set $Y = (\lambda^a)^{-1}(X)$ is an open neighborhood of u^a . Then $\{s^a\} \times Y$ is an open neighborhood of (s^a, u^a) which is disjoint from R_{m+1}^a , so (s^a, u^a) is not in the closure of R_{m+1}^a . By Lemma E.2, there are uncountably many $u^a \approx v^a$ and, therefore, $R_m^a \setminus \overline{R_{m+1}^a}$ is uncountable. *Q.E.D.*

PROOF OF THEOREM 10.1: By Proposition 7.1(ii), it suffices to assume that λ^a and λ^b are continuous. As such, Lemma F.1 gives that the set $R_0^a \setminus \overline{R_1^a}$ is uncountable. Then, by induction and Lemma F.2, for each m , the sets $R_{2m}^a \setminus \overline{R_{2m+1}^a}$ and $R_{2m+1}^b \setminus \overline{R_{2m+2}^b}$ are uncountable. Suppose that $(s^b, t^b) \in \bigcap_m R_m^b$. Then, for each m , we have that R_m^a is assumed under $\lambda^b(t^b)$ at some level $j(m)$. Moreover, the sequence $j(m)$ is nonincreasing. Then by Property 6.2 and the fact that each $R_{2m}^a \setminus \overline{R_{2m+1}^a}$ is uncountable, we have that each $j(2m + 1) < j(2m)$. But this contradicts the fact that $\lambda^b(t^b)$ has finite length. *Q.E.D.*

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