Use of a Canonical Hidden-Variable Space in Quantum Mechanics*

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Abstract. In Brandenburger and Keisler (2012b) we showed that, provided only that the measurement and outcome spaces in an experimental system are measure-theoretically separable, then there is a canonical hidden-variable space, namely the unit interval equipped with Lebesgue measure. Here, we use this result to establish a general relationship between two kinds of conditions on correlations in quantum systems: Bell locality (1964) and λ-independence on the one hand, and no signaling (Ghirardi, Rimini, and Weber (1980), Jordan (1983)) on the other hand.

"I rose the next morning, with Objective-Subjective and Subjective-Objective inextricably entangled together in my mind"[3]

1 Introduction

Among the most striking properties of quantum systems is that of **entanglement** — i.e., stronger-than-classical correlations — between particles that may be situated a large distance apart from each other. Bell (1964) famously proved that these correlations are indeed stronger-than-classical, but left open the question of just how strong they can be. A natural candidate for the answer to this question is that the correlations can be arbitrarily strong, provided they do not violate relativistic causality. Popescu and Rohrlich (1994) showed that this is false. There are correlations that respect relativistic causality and yet are stronger than can arise in any quantum system — they are **superquantum**.

The formal statements of these propositions rely on giving mathematical content to the concepts of classicality and relativistic causality. The first is captured via the condition of **Bell locality** (1964) combined with **λ-independence**, while the second is captured via the condition of **no signaling** (Ghirardi, Rimini, and Weber (1980), Jordan (1983)). In these terms, quantum correlations are a strict superset of correlations satisfying locality and λ-independence, and are a strict subset of correlations satisfying no signaling.

In particular, then, we know that the conjunction of locality and λ-independence is a strictly stronger condition on correlations than is no signaling. But, can we say more about the relationship?

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[3] *The Moonstone*, by Wilkie Collins, 1868

Locality itself is known to be equivalent to the conjunction of two conditions, namely **parameter independence** and **outcome independence** (Jarrett (1984), Shimony (1986)). In this chapter we give a result that relates no signaling to the conjunction of parameter independence and λ-independence.

Our result relates to work being done by the first author with Samson Abramsky and Andrei Savochkin, the purpose of which is to provide a justification for the no-signaling condition that does not involve an appeal to a different branch of physics (special relativity). The present result is technical in nature. It extends a result from Brandenburger and Yanofsky (2008), which is used in Abramsky, Brandenburger, and Savochkin (2013), from finite to infinite measurement spaces. It also relates to recent work by Colbeck and Renner (2011), (2012) on the issue of whether a subjective (or epistemic) vs. objective (or ontic) view of quantum states is tenable; see, especially their (2011p.4).

2 Preliminaries

Alice has a space of possible measurements, which is a measurable space $(Y_a, \mathcal{Y}_a)$, and a space of possible outcomes, which is a finite set X_a equipped with its power set, denoted $\mathcal{X}_a$. Likewise, Bob has a space of possible measurements, which is a measurable space $(Y_b, \mathcal{Y}_b)$, and a space of possible outcomes, which is a finite set X_b equipped with its power set, denoted $\mathcal{X}_b$. There is also a hidden-variable space, which is an unspecified measurable space $(\Lambda, \mathcal{L})$. We restrict attention to bipartite systems, but our result extends to multipartite systems. Write

$$(X, \mathcal{X}) = (X_a, \mathcal{X}_a) \otimes (X_b, \mathcal{X}_b), \tag{1}$$

$$(Y, \mathcal{Y}) = (Y_a, \mathcal{Y}_a) \otimes (Y_b, \mathcal{Y}_b). \tag{2}$$

Definition 1. *An **empirical model** is a probability measure e on $(X, \mathcal{X}) \otimes (Y, \mathcal{Y})$.*

Definition 2. *A **hidden-variable model** is a probability measure p on $(X, \mathcal{X}) \otimes (Y, \mathcal{Y}) \otimes (\Lambda, \mathcal{L})$.*

Definition 3. *We say that a hidden-variable model p **realizes** an empirical model e if $e = \mathrm{marg}_{X \times Y} p$. We say that two hidden-variable models, possibly with different hidden-variable spaces, are (**realization-**)**equivalent** if they realize the same empirical model.*

All definitions and notation parallel those in Section 3 of our 2012a paper. The reader should consult that paper for details and, in particular, for the notation for **conditional probability**, the definition of the **extension** of a probability measure, and the definition of the **fiber product** $p \otimes_Z q$ of two probability measures p and q over Z.

The key technique we use in proving our main result in the next section is the replacement of an arbitrary hidden-variable model with one where the hidden-variable space $(\Lambda, \mathcal{L})$ is the unit interval with the Borel subsets and $\mathrm{marg}_\Lambda p$ is Lebesgue measure. Theorem 5.1 in our (2012b) shows if the measurement and outcome spaces are countably generated, then this can always be done — in such a way that the two hidden-variable models are realization-equivalent and that various properties (parameter independence and λ-independence included) satisfied by the first model are again satisfied by the second model.

The next two definitions are taken from Section 4 of our 2012a paper.

Definition 4. *The hidden-variable model p satisfies **parameter independence** if for every $x_a \in X_a$ and $x_b \in X_b$ we have*

$$p[x_a || \mathcal{Y} \otimes \mathcal{L}] = p[x_a || \mathcal{Y}_a \otimes \mathcal{L}], \quad p[x_b || \mathcal{Y} \otimes \mathcal{L}] = p[x_b || \mathcal{Y}_b \otimes \mathcal{L}]. \tag{3}$$

In words, the probability of a particular outcome for Alice, if conditioned on Alice's choice of measurement and the value of the hidden variable, does not depend on Bob's choice of measurement; and vice versa, with Alice and Bob interchanged.

Definition 5. *The hidden-variable model p satisfies λ-**independence** if for every event $L \in \mathcal{L}$,*

$$p[L||\mathcal{Y}]_y = p(L). \tag{4}$$

This is an independence requirement between the hidden variable on the one hand, and the measurements chosen by Alice and Bob on the other hand. Whatever process determines the value of the hidden variable, this process does not influence the measurements Alice and Bob choose.

Next is the property of empirical models which we study.

Definition 6. *An empirical model e satisfies **no signaling** if for every $x_a \in X_a$ and $x_b \in X_b$ we have*

$$e[x_a||\mathcal{Y}] = e[x_a||\mathcal{Y}_a], \quad e[x_b||\mathcal{Y}] = e[x_b||\mathcal{Y}_b]. \tag{5}$$

In words, the probability of a particular outcome for Alice, if conditioned on Alice's choice of measurement, does not depend on Bob's choice of measurement; and vice versa, with Alice and Bob interchanged.

We will make use of the following notation:

$$e_a = \mathrm{marg}_{X_a \times Y_a} e, \quad e_b = \mathrm{marg}_{X_b \times Y_b} e, \tag{6}$$

$$s = \mathrm{marg}_Y e, \tag{7}$$

$$p_a = \mathrm{marg}_{X_a \times Y \times \Lambda} p, \quad p_b = \mathrm{marg}_{X_b \times Y \times \Lambda} p, \tag{8}$$

$$q_a = \mathrm{marg}_{X_a \times Y_a \times \Lambda} p, \quad q_b = \mathrm{marg}_{X_b \times Y_b \times \Lambda} p, \tag{9}$$

$$r = \mathrm{marg}_{Y \times \Lambda} p. \tag{10}$$

Lemma 1. *An empirical model e satisfies no signaling if and only if e is a common extension of the fiber products $e_a \otimes_{Y_a} s$ and $e_b \otimes_{Y_b} s$.*

Proof. By Lemma 3.6 in our 2012a paper.

3 The Result

Theorem 1. *Assume that the σ-algebra $\mathcal{Y}$ is countably generated. Then an empirical model e satisfies no signaling if and only if there is a hidden-variable model p which realizes e and satisfies parameter independence and λ-independence.*

Proof. First suppose e satisfies no signaling. We build the (trivial) hidden-variable model where Λ is a singleton. It is immediate that this model realizes e and satisfies parameter independence and λ-independence.

Now suppose that there is a hidden-variable model p which realizes e and satisfies parameter independence and λ-independence. By Lemma 1, we must show that e is an extension of the fiber product $e_a \otimes_{Y_a} s$.

By Theorem 5.1 in our 2012b paper, e is realized by a hidden-variable model p where $(\Lambda, \mathcal{L})$ is the unit interval with the Borel subsets, $\text{marg}_\Lambda p$ is Lebesgue measure, and p satisfies parameter independence and λ-independence. Let $\mathcal{L}^1, \mathcal{L}^2, \ldots$ be an increasing chain of finite algebras of sets whose union generates $\mathcal{L}$. By parameter independence, p_a is the fiber product $p_a = q_a \otimes_{Y_a \times \Lambda} r$.

For each n, let q_a^n and r^n be the restrictions of q_a and r to $\mathcal{X}_a \otimes \mathcal{Y}_a \otimes \mathcal{L}^n$ and $\mathcal{Y} \otimes \mathcal{L}^n$ respectively. In general, p will not be an extension of the fiber product $q_a^n \otimes_{Y_a \times \Lambda} r^n$. Our plan is to show that $q_a^n \otimes_{Y_a \times \Lambda} r^n$ is an extension of $e_a \otimes_{Y_a} s$, and converges to p_a as $n \to \infty$.

We first prove convergence. Fix an integer $k > 0$, and element $x_a \in X_a$, and sets $U \in \mathcal{Y}_a \otimes \mathcal{L}^k$ and $K_b \in \mathcal{Y}_b$. Then $q_a^n[x_a||\mathcal{Y}_a \otimes \mathcal{L}^n]$ is a uniformly bounded martingale with respect to the sequence of σ-algebras $\mathcal{Y}_a \otimes \mathcal{L}^n$, $n \geq k$. By the Martingale Convergence Theorem (Billingsley (1995, Theorem 35.5)), $q_a^n[x_a||\mathcal{Y}_a \otimes \mathcal{L}^n]$ converges to $q_a[x_a||\mathcal{Y}_a \otimes \mathcal{L}]$ p-almost everywhere. Similarly, for each $K_b \in \mathcal{Y}_b$, $r^n[K_b||\mathcal{Y}_a \otimes \mathcal{L}^n]$ converges to $r[K_b||\mathcal{Y}_a \otimes \mathcal{L}]$ p-almost everywhere. We have

$$(q_a^n \otimes_{Y_a \times \Lambda} r^n)(\{x_a\} \times U \times K_b) = \int_U q_a^n[x_a||\mathcal{Y}_a \otimes \mathcal{L}^n] \times r^n[K_b||\mathcal{Y}_a \otimes \mathcal{L}^n]\, dp \tag{11}$$

and

$$p_a(\{x_a\} \times U \times K_b) = \int_U q_a[x_a||\mathcal{Y}_a \otimes \mathcal{L}] \times r[K_b||\mathcal{Y}_a \otimes \mathcal{L}]\, dp. \tag{12}$$

Moreover, as $n \to \infty$,

$$q_a^n[x_a||\mathcal{Y}_a \otimes \mathcal{L}^n] \times r^n[K_b||\mathcal{Y}_a \otimes \mathcal{L}^n] \to q_a[x_a||\mathcal{Y}_a \otimes \mathcal{L}] \times r[K_b||\mathcal{Y}_a \otimes \mathcal{L}] \tag{13}$$

p-almost everywhere. By Fatou's Lemma (Billingsley (1995, Theorem 16.3)),

$$\int_U q_a^n[x_a||\mathcal{Y}_a \otimes \mathcal{L}^n] \times r^n[K_b||\mathcal{Y}_a \otimes \mathcal{L}^n]\, dp \to \int_U q_a[x_a||\mathcal{Y}_a \otimes \mathcal{L}] \times r[K_b||\mathcal{Y}_a \otimes \mathcal{L}]\, dp. \tag{14}$$

Therefore

$$(q_a^n \otimes_{Y_a \times \Lambda} r^n)(\{x_a\} \times U \times K_b) \to p_a(\{x_a\} \times U \times K_b). \tag{15}$$

It follows that for each $x^a \in X_a, K_a \in \mathcal{Y}_a$, and $K_b \in \mathcal{Y}_b$,

$$(q_a^n \otimes_{Y_a \times \Lambda} r^n)(\{x_a\} \times K_a \times K_b) \to p_a(\{x_a\} \times K_a \times K_b) = e(\{x_a\} \times K_a \times K_b). \tag{16}$$

We next prove that for each n, $q_a^n \otimes_{Y_a \times \Lambda} r^n$ is an extension of $e_a \otimes_{Y_a} s$. Let $\mathcal{A}^n$ be the set of all atoms of $\mathcal{L}^n$ of positive Lebesgue measure. Then $\mathcal{A}^n$ is a finite collection of pairwise disjoint subsets of Λ whose union has Lebesgue measure 1. Let $u = q_a^n \otimes_{Y_a \times \Lambda} r^n$. By Lemma 3.6 in our 2012a paper,

$$u[x_a||\mathcal{Y} \otimes \mathcal{L}^n] = u[x_a||\mathcal{Y}_a \otimes \mathcal{L}^n]. \tag{17}$$

The conditional probability $u[x_a||\mathcal{Y} \otimes \mathcal{L}^n]_{(y,\lambda)}$ depends only on y and the atom $A \in \mathcal{A}^n$ that contains λ, so we may write

$$u[x_a||\mathcal{Y} \otimes \mathcal{L}^n]_{(y,\lambda)} = u[x_a||\mathcal{Y} \otimes \mathcal{L}^n]_{(y,A)} \tag{18}$$

whenever $\lambda \in A \in \mathcal{A}^n$. We have

$$u[x_a||\mathcal{Y}]_y = \sum\nolimits_{A \in \mathcal{A}^n} u[x_a||\mathcal{Y} \otimes \mathcal{L}^n]_{(y,A)} \times p[A||\mathcal{Y}]_y. \tag{19}$$

A similar computation holds with $\mathcal{Y}_a$ in place of $\mathcal{Y}$. Since p satisfies λ-independence,

$$p[A||\mathcal{Y}]_y = p(A) = p[A||\mathcal{Y}_a]_y \tag{20}$$

for each $A \in \mathcal{A}^n$ and $y \in Y$. Therefore

$$u[x_a||\mathcal{Y}] = u[x_a||\mathcal{Y}_a]. \tag{21}$$

Since q_a^n is an extension of e_a, and r^n is an extension of s, we have from Lemma 3.6 in our 2012a paper that $u = q_a^n \otimes_{Y_a \times \Lambda} r^n$ is an extension of $e_a \otimes_{Y_a} s$. Thus

$$(e_a \otimes_{Y_a} s)(\{x_a\} \times K_a \times K_b) \tag{22}$$

is a constant sequence that converges to $e(\{x_a\} \times K_a \times K_b)$, and hence

$$(e_a \otimes_{Y_a} s)(\{x_a\} \times K_a \times K_b) = e(\{x_a\} \times K_a \times K_b) \tag{23}$$

for all $x_a \in X_a, K_a \in \mathcal{Y}_a$, and $K_b \in \mathcal{Y}_b$. This shows that e is an extension of $e_a \otimes_{Y_a} s$. A similar argument holds for b in place of a, so e satisfies no signaling by Lemma 1 above.

References

[2013] Abramsky, S., A. Brandenburger, and A. Savochkin, "No-Signalling is Equivalent to Free Choice of Measurements," 2013.

[1964] Bell, J., "On the Einstein-Podolsky-Rosen Paradox," *Physics*, 1, 1964, 195-200.

[1995] Billingsley, P., *Probability and Measure*, 3rd edition, Wiley, 1995.

[2012a] Brandenburger, A., and H.J. Keisler, "Fiber Products of Measures and Quantum Foundations," 2012. Forthcoming in Chubb, J., A. Eskandarian, and V. Harizanov (eds.), *Logic & Algebraic Structures in Quantum Computing & Information*, Association for Symbolic Logic/Cambridge University Press.

[2012b] Brandenburger, A., and H.J. Keisler, "A Canonical Hidden-Variable Space," 2012. Available at `http://www.stern.nyu.edu/~abranden` and `http://www.math.wisc.edu/~keisler`.

[2008] Brandenburger, A., and N. Yanofsky, "A Classification of Hidden-Variable Properties," *Journal of Physics A: Mathematical and Theoretical*, 41, 2008, 425302.

[2011] Colbeck, R., and R. Renner, "No Extension of Quantum Theory Can Have Improved Predictive Power," *Nature Communications*, 2, 2011, 411.

[2012] Colbeck, R., and R. Renner, "Is a System's Wave Function in One-to-One Correspondence with Its Elements of Reality?" *Physical Review Letters*, 108, 2012, 150402.

[1980] Ghirardi, G, A. Rimini, and T. Weber, "A General Argument Against Superluminal Transmission Through the Quantum Mechanical Measurement Process," *Lettere Al Nuovo Cimento (1971-1985)*, 27, 1980, 293-298.

[1984] Jarrett, J., "On the Physical Significance of the Locality Conditions in the Bell Arguments," *Noûs*, 18, 1984, 569-589.

[1983] Jordan, T., "Quantum Correlations Do Not Transmit Signals," *Physics Letters A*, 94, 1983, 264.

[1994] Popescu, S., and D. Rohrlich, "Quantum Nonlocality as an Axiom," *Foundations of Physics*, 24, 1994, 379-385.

[1986] Shimony, A., "Events and Processes in the Quantum World," in Penrose, R., and C. Isham (eds.), *Quantum Concepts in Space and Time*, Oxford University Press, 1986, 182-203.