Signed Subjective Expected Utility*

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Abstract

In the world of subjective probability, there is no a priori reason why probabilities — interpreted as a willingness-to-bet—should necessarily lie in the interval $[0, 1]$. We weaken the Monotonicity axiom in classical subjective expected utility (Anscombe and Aumann, 1963) to obtain a representation of preferences in terms of an affine utility function and a signed (subjective) probability measure on states. We decompose this probability measure into a non-negative probability measure ("probability") and an additive set function on states which sums to 0 ("valence"). States with positive (resp. negative) valence are attractive (resp. aversive) for the decision maker. We show how our decision theory can resolve the paradoxes of the conjunction effect (Tversky and Kahneman, 1982, 1983), the co-existence of insurance and betting (Friedman and Savage, 1948), and the choice of dominated strategies in strategy-proof mechanisms (Hassidim et al., 2016). We also comment on the possible application of our theory to signed-probability representations of quantum mechanics (originating in Wigner, 1932).

Keywords: Signed probabilities, non-monotonicity, indifference substitution, valence, attractive and aversive states

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1 Introduction

In a world of objective probability, where probabilities are an idealization of empirical frequencies, the probability of any event must lie between 0 and 1. Interestingly, this property of probability was carried over to the world of subjective probability by the pioneers, such as de Finetti (1937), Ramsey (1926), and Savage (1954), without change.

In this paper, we re-examine the world of subjective probability, once one allows that there is no fundamental reason why a subjective assessment—interpreted, as usual, as a willingness-to-bet—should lie in the same 0 to 1 range. That is, we allow for the possibility both of events with subjective probability less than 0 and of events with subjective probability greater than 1. In the language of probability theory, we admit signed probabilities.

Clearly, these probabilities cannot be equated to frequencies, in the way subjective probabilities are sometimes thought of as aligned with objective quantities. To gain some insight into their interpretation, consider the following scenario. A decision maker (DM) is asked to choose between four bets. Each bet pays off \$1 in the stated event and \$0 otherwise. The first bet is that the Italian tennis star Jannik Sinner wins the match. The second bet is that Sinner loses the first set. The third bet is that Sinner loses the first set but goes on to win the match. The fourth bet is that Sinner wins the first set but goes on to lose the match. We suggest that a DM who is a big Sinner fan might express a preference for the first bet over the third bet over the second bet over the fourth bet. Indeed, Tversky and Kahneman (1982, 1983) introduced the above scenario without betting (and with Björn Borg as the then tennis great) and found what they termed the conjunction fallacy: Some subjects viewed the event that Sinner loses the first set but goes on to win the match as more probable than the event that Sinner loses the first set, even though the latter event strictly contains the former.

The betting scenario we present is obviously linked to the conjunction fallacy, with two differences. First, we work in terms of explicit bets rather than probability assessments. This is in line with the view of subjective probability as a representation of willingness-to-bet. (See Sides et al. (2002) for consistent evidence for the conjunction fallacy across several betting scenarios.)

Second, we suggest that the DM's behavior is not based on a failure to process properly the likelihoods of joint events, but is explainable via an intrinsic "valence"—to use a psychological term—on states of the world. Specifically, the DM attributes a valence to states, with some states viewed as attractive and other states viewed as aversive.[1] In the scenario, the DM views the state in which Sinner loses the first set and then loses the overall match as aversive—because the DM is a Sinner fan. Accordingly, in our decision model, the DM assigns a negative net probability to

[1] From `https://dictionary.apa.org/valence`: Valence [is] the subjective value of an event, object, person, or other entity in the life space of the individual. An entity that attracts the individual has positive valence whereas one that repels has negative valence.

this state, which allows for the stated preference pattern. The net probability of each state is the sum of an ordinary subjective probability (lying in in $[0, 1]$) of the state and the valence of the state, which may be positive (for an attractive state) or negative (for an aversive state).

There are other contexts in which the conjunction fallacy is evidenced, such as scenarios requiring simple physical reasoning (Ludwin-Peery et al., 2020). We do not claim that our model is a good explanation of all such occurrences, although it is possible that there are other "less psychological" interpretations of our valence function.

Our DM's preference relation is inconsistent with Anscombe and Aumann (1963) subjective expected utility (SEU). Indeed, it is inconsistent with any model satisfying the well-known axiom of Monotonicity. This axiom states that if an act g yields weakly more desirable outcomes than another act f, across all possible states of the world, then g should be weakly preferred to f. In our example, let g be the second bet and f the third bet. Then the outcomes are: \$0 vs. \$0 if Sinner wins the first set; \$1 vs. \$1 if Sinner loses the first set and wins the match; and \$1 vs. \$0 if Sinner loses the first set and loses the match. If the third bet is preferred to the second bet, Monotonicity fails.

We replace Monotonicity with a weaker axiom we call Indifference Substitution, which requires only that if acts f and g yield equally desirable outcomes, across all possible states of the world, then f and g should be indifferent. (This axiom appears in Grant and Polak, 2011, where it is called Substitution.) Our main result shows that the standard axioms for subjective expected utility, with Monotonicity replaced by Indifference Substitution, characterize a representation of preferences by a standard von Neumann-Morgenstern utility on outcomes and a signed probability measure on states. Recall that a signed probability measure is an additive set function that assigns measure 1 to the overall state space, but may assign measure greater than 1 or less than 0 to some events in the space. We refer to our representation as "signed subjective expected utility" (SSEU). Paralleling standard subjective expected utility, the utility function in our representation is unique up to affine transformations and the (signed) probability measure is unique. Notice that while our representation can be seen as a particular kind of state-dependent utility, it comes with the advantage that its uniqueness properties allow us to separate and uniquely identify utility and subjective probability.

A signed probability measure ν on a finite state space S can be uniquely decomposed into an ordinary (non-negative) probability measure p and a second additive set function γ satisfying $\gamma(S) = 0$.[2] Formally: $\nu = p + \gamma$. We already referred to the function γ as the valence, in the psychological sense, of the DM. Ramsey (1926) observed the possibility that states might hold an intrinsic desire (or the opposite) for a DM when he wrote:

> "[T]he propositions ... which are used as conditions in the options offered may be such

[2] Note that this is not the usual Jordan decomposition of a signed measure.

that their truth or falsity is an object of desire to the subject. This will be found to complicate the problem, and we have to assume that there are propositions for which this is not the case, which we shall call ethically neutral."

In Ramsey's phrasing, our set-up allows for propositions (states) that are not ethically neutral, but, instead, may carry an intrinsic positive (attractive) or negative (aversive) valence. Our valence function is the DM's measure of "ethical non-neutrality."

We employ our SSEU model to offer new resolutions of several well-known anomalies in decision theory. Our first example is a look at the conjunction fallacy (Tversky and Kahneman, 1982, 1983) from the point of view of our decomposition $\nu = p + \gamma$. We establish and interpret a sufficient condition on the ordinary probability measure p and valence γ for the conjunction-fallacy effect to arise. Our second example goes back to the classic Friedman and Savage (1948) paradox of the co-existence of insurance and betting behavior. These authors offered a resolution that depends on changing risk attitudes with changing wealth levels. By contrast, our resolution operates at a single wealth level—so that "co-existence" becomes truly the simultaneous purchase of insurance and a lottery ticket—and works if the risk-averse DM has a sufficiently positive valence for winning the lottery. Our third application of SSEU provides an explanation of empirical evidence that some individuals choose dominated strategies in strategy-proof mechanisms, such as the well-known deferred acceptance algorithm (Hassidim et al., 2016; Shorrer and Sóvágó, 2023; Dreyfuss et al., 2022). The key to our resolution is that the DM has a sufficiently negative valence for not being matched with their first choice.

There is a possible objection to our theory, which is that a DM following SSEU might be susceptible to a "Dutch Book," and a consequent money pump (de Finetti, 1937). If the DM's willingness-to-bet on event E is $\$\nu(E)$, where $\nu(E) > 1$, then a bettor who does not share this assessment will be able to obtain a sure win, by selling the DM a bet with prize \$1 if E obtains and \$0 otherwise—at a price of $\$(\nu(E) - \epsilon)$ (where $\epsilon < \nu(E) - 1$). But, while this might be feasible as a one-time transaction, it is not clear to us that it will be repeatable. After the first transaction, the DM might well change valence so that $\nu(E) \leq 1$ for subsequent bets. Money pumps can occur only in the presence of unchanging preferences over the course of the pump, and we see no reason to assume that valence might not vary with the DM's betting history. Even for the one-time bet, the sure loss looks troublesome only from an outsider's perspective (assuming the outsider follows classical SEU). From the DM's perspective, the positive valence attached to E implies a (non-monetary) welfare gain from betting on E.

We next mention a very different motivation for our new decision theory with signed probabilities. While quantum mechanics (QM) is usually formulated in terms of Hilbert spaces and Hermitian operators (Sakurai and Napolitano, 2020, is a standard introduction), there is a fully equivalent formulation in terms of phase space (analogous to the state space of decision theory) and signed probabilities. This equivalence was first formulated by Wigner (1932), and the use of signed probabilities in QM was promoted by Dirac (1942) and Feynman (1987). Today, phase-space QM is routinely employed in e.g. quantum optics (Kenfack and Życzkowski, 2004). Since the probabilities in QM are objective, only unobserved events can receive probability outside the interval $[0, 1]$. But, the subtlety is that even though the appearance of non-classical probability is restricted in this fashion, this is sufficient to allow observed behavior, such as quantum entanglement (see Horodecki et al., 2009), that is impossible in the classical physical world.

We believe that our SSEU theory may find useful applications to QM. As one possibility, our novel decomposition $\nu = p + \gamma$ of signed probabilities suggests a new measure of the degree of non-classicality of a quantum system based on the quantity $\sum_{s \in S} |\gamma(s)|$ (or some other norm of the valence). See Camillo and Cervantes (2024) for a review of various measures of non-classicality (called contextuality) in QM.

Brandenburger et al. (2024a) is another application of signed probabilities—to an examination of the status of the Agreement Theorem (Aumann, 1976) of classical epistemics in a non-classical setting. That work was motivation for the development of a fully axiomatized signed decision theory as in the current paper. In their examination, Brandenburger et al. (2024a) find that conditioning becomes much more complex in a world of signed probabilities, fundamentally because an event of probability 0 may contain a sub-event of strictly positive (or negative) probability. In a companion paper to the current one (Brandenburger et al., 2024b), we develop a theory of "signed conditional probability spaces," extending the classical theory of conditional probability spaces due to Rényi (1955).

Perea (2022) develops a decision theory where the axioms require the DM to entertain signed as well as standard probabilities on the set of states, although the final representation is standard. Ke and Zhao (2023) provide new representation results for decision making under ambiguity, including one where the representation involves a signed probability measure over states.

The organization of the rest of the paper is as follows. Section 2 lays out our decision framework, defines valence, and presents our SSEU representation. Section 3 defines our four axioms—Weak Order, Independence, Archimedean Property, and Indifference Substitution—and states our representation theorem. It also offers three definitions of null events, which coincide in the case of ordinary probability but are distinct in our setting. Section 4 contains our resolutions of the conjunction-fallacy effect, the paradox of the coexistence of insurance and betting behavior, and the puzzle of choice of dominated strategies in strategy-proof mechanisms. It also examines the comparative statics of valence. Section 5 extends our decision theory by weakening Independence to Comonotonic Independence (Schmeidler, 1989) to obtain a signed Choquet representation. This decision theory naturally accommodates the experimental results in Schneider and Schonger (2019), where subjects jointly violate Monotonicity and Independence. Except where indicated otherwise, the proofs of the results are found in the appendices.

2 Preliminaries

2.1 Choice Setting

Consider a finite set S of *states of the world* and a set X of *consequences*. A subset $E \subseteq S$ is called an *event*. We denote by $\mathcal{F}$ the set of all functions (called *acts*) $f : S \to X$.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the *constant* act such that $x(s) = x$ for all $s \in S$. With the usual slight abuse of notation, we thus identify X with the subset of the constant acts in $\mathcal{F}$. If $f, g \in \mathcal{F}$, and an event $E \subseteq S$, we denote by $fEg \in \mathcal{F}$ the act that yields $f(s)$ if $s \in E$ and $g(s)$ if $s \notin E$. Given $f \in \mathcal{F}$ and $u : X \to \mathbb{R}$, $u(f)$ denotes the function $s \mapsto u(f(s))$.

We assume additionally that X is a convex subset of a vector space. We can define for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act $\alpha f + (1 - \alpha)g \in \mathcal{F}$, which yields $\alpha f(s) + (1 - \alpha)g(s) \in X$ for every $s \in S$.

2.2 Preferences

We model the decision maker's (DM's) preferences on $\mathcal{F}$ by a binary relation $\succcurlyeq$. As usual, $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succcurlyeq$. If $f \in \mathcal{F}$, an element $x_f \in X$ is a certainty equivalent for f if $f \sim x_f$.

A signed probability measure is a function $\nu : S \to \mathbb{R}$ such that $\sum_{s \in S} \nu(s) = 1$. Given $E \subseteq S$, set $\nu(E) = \sum_{s \in E} \nu(s)$. The set $\Delta(S)$ denotes the set of all signed probability measures on S, that is

$$\Delta(S) = \left\{ \nu : S \to \mathbb{R} : \sum_{s \in S} \nu(s) = 1 \right\}.$$

We denote by $\Delta^+(S)$ the set of ordinary (non-negative) probability measures on S, i.e.

$$\Delta^+(S) = \left\{ \nu : S \to [0, \infty) : \sum_{s \in S} \nu(s) = 1 \right\}.$$

Given $\phi : S \to \mathbb{R}$ and $\nu \in \Delta(S)$, set

$$\int_S \phi d\nu := \sum_{s \in S} \phi(s)\nu(s).$$

2.3 The Model

We start by introducing a set function, called the *valence*, that quantifies the attractiveness or aversiveness of an event.

Definition 1. A function $\gamma : 2^S \to \mathbb{R}$ is called a **valence** if

1. For all $E, F \subseteq S$, if $E \cap F = \emptyset$, then $\gamma(E \cup F) = \gamma(E) + \gamma(F)$;

2. $\gamma(E) + \gamma(E^c) = 0$.

Note that Conditions 1 and 2 imply $\gamma(S) = 0$, $\gamma(\emptyset) = 0$, and $\gamma(A) = \sum_{s\in A}\gamma(s)$. Condition 1 says that the valence of an event depends on the valencse of the states comprising the event (we relax this property in Section 5). Condition 2 says that if an event has a positive valence, its complement has a negative valence (again, we relax this assumption in Section 5). Moreover, the valence of the entire state space is 0.

Definition 2. We say that (u, p, γ) is a **valence representation of** $\succcurlyeq$ if $u : X \to \mathbb{R}$ is an affine function, $p \in \Delta^+(S)$, and γ is a valence such that

$$f \succcurlyeq g \iff \int_S u(f)dp + \int_S u(f)d\gamma \geq \int_S u(g)dp + \int_S u(g)d\gamma.$$

If one defines the set function $\nu = p + \gamma$, the properties of the probability p and of the valence γ ensure that ν is a signed measure with $\nu(S) = 1$. Indeed, both p and γ are additive over disjoint unions and $\nu(S) = p(S) + \gamma(S) = 1$, so that $\nu \in \Delta(S)$. Therefore, any valence representation can be rewritten as an SSEU representation, as follows.

Definition 3. We say that (u, ν) is a **signed subjective expected utility** (**SSEU**) representation of $\succcurlyeq$ if $u : X \to \mathbb{R}$ is an affine function and $\nu \in \Delta(S)$ are such that

$$f \succcurlyeq g \iff \int_S u(f)d\nu \geq \int_S u(g)d\nu.$$

We now show that any SSEU representation has a valence representation.

Proposition 1. *The preferences $\succcurlyeq$ have an SSEU representation if and only if they has a valence representation.*

We sketch the proof and leave the details to Appendix B. We have already shown sufficiency. For necessity, observe that if we define $\nu^+, \nu^- \in \mathbb{R}^S$ by

$$\nu^+(s) = \max\{0, \nu(s)\} \text{ and } \nu^-(s) = -\max\{0, -\nu(s)\},$$

for every $s \in S$, we obtain the Jordan decomposition

$$\nu(s) = \nu^+(s) - \nu^-(s) \text{ for every } s \in S.$$

This decomposition can in turn be written as (see Proposition 3 in Appendix B)

$$\nu^+(s) - \nu^-(s) = (1 + b)p^+(s) - bp^-(s),$$

for every $s \in S$, where p^+, $p^- \in \Delta^+(S)$ and $b \geq 0$. A valence representation follows once we let $p = p^+$ and $\gamma = b(p^+ - p^-)$. We call γ, which we extend to all subsets of S using the properties of a valence function, the **minimal valence** of ν. Likewise, we call the above decomposition the **minimal decomposition** of ν. The terminology is due to the fact that multiple decompositions of the signed measure ν are possible, and therefore multiple valence representations, but the one obtained above is the unique one which minimizes the (absolute value of the) valence function event-by-event, and thus provides the most "classical" description of the DM's choices.

To illustrate valence representations, consider a bet xEy with $x \succcurlyeq y$. Its SSEU value is

$$V(xEy) = \underbrace{p(E)u(x) + (1 - p(E))u(y)}_{\text{SEU component}} + \underbrace{\gamma(E)(u(x) - u(y))}_{\text{valence component}}.$$

The value of betting on E is the sum of a classical SEU component and a non-classical valence component. Since $x \succcurlyeq y$, $u(x) \geq u(y)$, so that a positive (resp. negative) valence $\gamma(E)$ will increase (resp. decrease) the overall value of the bet xEy. In particular, if $u(x) = 1$ and $u(y) = 0$, then $V(xEy) = p(E) + \gamma(E)$, highlighting the decomposition of the DM's willingness-to-bet on E into likelihood and valence components.

It is important to observe that an event which commands a willingness-to-bet larger than 1 is not necessarily perceived by the DM as a "sure event," i.e., as an event with likelihood equal to 1. Clearly, the DM can assess $p(E) + \gamma(E) > 1$ while $p(E) < 1$. An analogous remark applies to events which command a negative willingness-to-bet.

Next, we provide a few examples of SSEU representations.

Example 1. Suppose there is only one aversive state s^*, so that $\gamma(s) = b(p^+(s) - \delta_{s^*})$. Then

$$V(f) = \int_S u(f)dp^+(s) + b\left(\int_S u(f)dp^+(s) - u(f(s^*))\right).$$

Consider the bet xEy for some event E with $s^* \in E^c$. Then

$$V(xEy) = p^+(E)u(x) + (1 - p^+(E))u(y) + bp^+(E)(u(x) - u(y)),$$

and the willingness-to-bet on E is $\nu(E) = (1 + b)p^+(E)$. For general acts, if $f \succcurlyeq g$, then

$$\int_S u(f)dp^+ - \int_S u(g)dp^+ \geq \frac{b}{1+b}(u(f(s^*)) - u(g(s^*)).$$

Suppose $u(f(s^*)) - u(g(s^*)) \geq 0$. Then a preference for f over g implies that the SEU component of the value of f is sufficiently larger than that of g to override the negative value of obtaining more utility in the aversive state. If $u(f(s^*)) - u(g(s^*)) \leq 0$, a preference for f over g can obtain even if the SEU component of the value of g is strictly larger than that of f. △

Example 2. Paralleling Example 1, suppose that there is only one attractive state s^*, so that $\gamma(s) = b(\delta_{s^*} - p^-(s))$. Then

$$V(f) = u(f(s^*)) + b\left(u(f(s^*)) - \int_S u(f(s))dp^-\right).$$

Consider the bet xEy for some E with $s^* \in E$. Then

$$V(xEy) = u(x) + b(1 - p^-(E))(u(x) - u(y)),$$

and the willingness-to-bet on E is $\nu(E) = 1 + b(1 - p^-(E))$. For general acts, if $f \succcurlyeq g$, then

$$u(f(s^*)) - u(g(s^*)) \geq \frac{b}{1+b}\left(\int_S u(f)dp^- - \int_S u(g)dp^-\right). \quad \triangle$$

Example 3. Let $\gamma(s) = k(p(s) - 1/|S|)$ for some $k \geq 0$ and $p \in \Delta^+(S)$. Then

$$V(f) = \mathbb{E}_{p^+}[u(f)] + k\left(\sum_{s\in S}(p(s)u(f(s)) - \frac{1}{|S|}\right).$$

There is positive (resp. negative) valence if a state is more (resp. less) likely than the uniform case. The willingness-to-bet on an event E is $\nu(E) = p^+(E) + k\,(p^+(E) - |E|/|S|)$. $\triangle$

Example 4. In the spirit of the Radon-Nikodym theorem, consider a non-minimal valence given by $\gamma(s) = p^+(s) - \phi(s)p^+(s)$ for some function $\phi : S \to \mathbb{R}$ with $\mathbb{E}_{p^+}[\phi] = 1$. Then

$$V(f) = \mathbb{E}_{p^+}[u(f)] + b\Big(\mathbb{E}_{p^+}[u(f)] - \mathbb{E}_{p^+}[u(f)\phi]\Big). \quad \triangle$$

3 Axiomatic Characterization

This section introduces the axioms characterizing our SSEU model. Start with the standard Anscombe-Aumann axioms that characterize SEU.

Axiom 1 (Weak Order - WO)**.** $\succcurlyeq$ is complete and transitive. Moreover, there exist $f, g \in \mathcal{F}$ such that $f \succ g$.

Axiom 2 (Independence - I)**.** If $f, g, h \in \mathcal{F}$ and $\gamma \in (0, 1]$, $f \succcurlyeq g$ implies $\gamma f + (1 - \gamma)h \succcurlyeq \gamma g + (1 - \gamma)h$.

Axiom 3 (Archimedean - A)**.** If $f, g, h \in \mathcal{F}$ and $f \succ g \succ h$, there are $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h$.

Axiom 4 (Monotonicity - M)**.** For every $f, g \in \mathcal{F}$, $f(s) \succcurlyeq g(s)$ for every $s \in S \implies f \succcurlyeq g$.

Recalling the discussion in the Introduction, we weaken Monotonicity as follows[3]

Axiom 5 (Indifference Substitution - IS)**.** For every $f, g \in \mathcal{F}$, $f(s) \sim g(s)$ for every $s \in S \implies f \sim g$.

According to IS, two acts that yield equivalent payoffs in all states must be deemed indifferent.[4] To illustrate the weaker scope of IS compared to Monotonicity, consider two payoffs $x \succ y$. Under

[3]One potential explanation for violation of Monotonicity is that this entails a subtle form of "state independence" of preferences, or weak separability, which may be overly restrictive. Monotonicity implies that, if $xEf \succ yEf$ for some event E and some act f, then $xFg \succcurlyeq yFg$ for all events F and acts g. In particular, if $x \succcurlyeq y$ then $xEf \succcurlyeq yEf$ for all events E and all acts f.

[4]This axiom appeared in Grant and Polak (2011), where it is called Substitution.

Monotonicity, it is necessary that $x \succcurlyeq xEy$ for all events E. However, IS allows for the possibility of $xEy \succ x$. This preference can be rationalized when the event E is attractive (i.e., it has a positive valence).

Replacing Monotonicity with Indifference Substitution characterizes our SSEU representation.

Theorem 1. *A binary relation $\succcurlyeq$ satisfies axioms WO, I, A, and IS if and only if there exists a non-constant affine function $u : X \to \mathbb{R}$ and a signed probability measure $\nu \in \Delta(S)$ such that (u, ν) is a SSEU representation of $\succcurlyeq$. Moreover, if (u', ν') is another SSEU representation of $\succcurlyeq$, then there exist $a, b \in \mathbb{R}$ with $a > 0$ and $u'(x) = au(x) + b$, and $\nu' = \nu$.*

By Proposition 1, the axioms in Theorem 1 are necessary and sufficient to obtain a Valence representation of $\succcurlyeq$. The next immediate result shows that Monotonicity implies the existence of a representation of $\succcurlyeq$ in which all events have ”zero” valence.

Corollary 1. *A binary relation $\succcurlyeq$ satisfies axioms WO, I, A, and M if and only if it has a valence representation (u, p, γ) where $\gamma(E) = 0$ for all $E \subseteq S$.*

This corollary follows directly by taking γ to be the minimal valence. By Monotonicity, the SSEU representation is an SEU representation, and $\nu = p^+$, implying $b = 0$.

3.1 Null Events

When a DM entertains signed probabilities, an event can be null but contain states that have non-zero probability. It therefore makes sense to posit that updating should in some way preserve beliefs over these internal states. We consider in particular the following distinct notions of null and non-null event.[5]

Definition 4. An event $E \subseteq S$ is $\succcurlyeq$**-null** if $xEy \sim y$ for some $x \not\sim y$; it is $\succcurlyeq$**-non-null** otherwise. An event $E \subseteq S$ is $\succcurlyeq$**-completely null** if every $F \subseteq E$ is $\succcurlyeq$-null. An event $E \subseteq S$ is $\succcurlyeq$**-classically null** if it there is no $F \subseteq E$ such that $xFy \succ y$ for some $x \succ y$.

The preceding definitions above coincide under SEU, but differ under SSEU. This can be seen from the following characterizations in terms of the minimal decomposition (p, γ).

Theorem 2. *Fix a preference relation $\succcurlyeq$ with an SSEU representation (u, ν) where u is non-constant, and let (p, γ) be the minimal decomposition of ν. An event E is $\succcurlyeq$-null if and only if $p(E) + \gamma(E) = 0$. An event is E $\succcurlyeq$-completely null if and only if $p(F) + \gamma(F) = 0$ for all $F \subseteq E$. An event E is $\succcurlyeq$-classically null if and only if $p(E) = 0$.*

[5] We are grateful to Miklós Pintér for suggesting the definition of a completely null event.

4 Applications

4.1 Conjunction Fallacy

Individuals exhibit the conjunction fallacy when they consider the likelihood of a conjunction of two events to be larger than the likelihood of one of the constituent events (Tversky and Kahneman, 1982, 1983). For example, consider the four events:

- E_1 = Sinner will win the match
- E_2 = Sinner will lose the first set
- E_3 = Sinner will lose the first set but win the match
- E_4 = Sinner will win the first set but lose the match

In a classic experiment due to Tversky and Kahneman (1983),[6] subjects ranked (on average) the event E_1 more likely than event E_3, event E_3 more likely than E_2, and event E_4 as the least likely of all. The conjunction fallacy is evidenced because event E_3 is the conjunction of events E_1 and E_2 (i.e., $E_3 = E_1 \cap E_2$), so that its likelihood should not, in a classical analysis, be larger than that of E_2.

Formally, a DM exhibits the conjunction fallacy if $\nu(E_3) > \nu(E_2)$. Our SSEU model can rationalize this relationship if the intrinsic aversiveness of the event E_2, where Sinner loses the first set is sufficiently higher than that of the event E_3. The following fact is immediate.

Proposition 2. *If $\gamma(E_3) - \gamma(E_2) > p(E_2) - p(E_3)$ then $\nu(E_3) > \nu(E_2)$.*

If, in addition to the condition of Proposition 2, we have $\gamma(E_1) = \gamma(E_1 \cap E_2) = \gamma(E_3)$, then $\nu(E_1) > \nu(E_1 \cap E_2) = \nu(E_3) > \nu(E_2)$. Next, consider the valence defined in Example 1, viz., $\gamma(s) = b(p^+(s) - \delta_{s^*})$. Suppose that the aversive state $s^* \in E_2 \backslash E_1$. Then

$$\nu(E_1) = p^+(E_1) + b(p^+(E_1) - \sum_{s \in E_1} \delta_{s^*}) = (1+b)p^+(E_1),$$
$$\nu(E_2) = p^+(E_2) + b(p^+(E_2) - \sum_{s \in E_2} \delta_{s^*}) = (1+b)p^+(E_2) - b,$$
$$\nu(E_3) = \nu(E_1 \cap E_2) = p^+(E_1 \cap E_2) + b(p^+(E_1 \cap E_2) - \sum_{s \in E_1 \cap E_2} \delta_{s^*}) = (1+b)p^+(E_1 \cap E_2).$$

Observe that we find $\nu(E_1) > \nu(E_1 \cap E_2) = \nu(E_3)$, and $\nu(E_3) > \nu(E_2)$ if and only if $p^+(E_2) - p^+(E_3) < b/(1+b)$.

[6] Involving then tennis great Björn Borg.

4.2 Coexistence of Insurance and Gambling

Standard expected utility theory with a concave or convex utility function struggles to explain why individuals simultaneously buy insurance and lottery tickets. The firsrt purchase reflects risk aversion, while the second reflects risk seeking. Friedman and Savage (1948) suggests that risk attitudes can vary with wealth levels, so that insurance and gambling can coexist in a "sequential" purchases. We propose an alternative explanation based on valence that does not require sequentiality of purchases.[7]

Loss	yes	no
	$x - \ell$	x

Table 1: Act f_1 - No insurance, No ticket

An insurance policy covers a loss $\ell \geq 0$ if the event L occurs. The premium is denoted by q and we assume that $0 \leq \nu(L) \leq 1$ and that $q = \nu(L)\ell$. A lottery pays $W \geq 0$ if the event E occurs and 0 otherwise. The price π of the lottery ticket is $0 \leq \pi \leq W$. The DM compares four actions.[8] The first option is to choose to purchase neither the insurance nor the lottery ticket. Table 1 shows the associated payoffs. The SSEU of the first act is

$$V(f_1) = (p(L) + \gamma(L))u(x - \ell) + (1 - p(L) - \gamma(L))u(x).$$

The second option is to buy the insurance but not the lottery ticket. Table 2 shows the associated payoffs.

Loss	yes	no
	$x - q$	$x - q$

Table 2: Act f_2 - Yes insurance, No ticket

The SSEU of the second act is

$$V(f_2) = u(x - q).$$

The third option is to buy the insurance but not the lottery ticket. Table 3 shows the associated payoffs.

[7]If the price of a lottery ticket is small enough, the "sequential" explanation would require a large local change in the curvature of the utility function.

[8]In this section, we make the assumption that X is the set of simple lotteries over $\mathbb{R}$, that is lotteries with finitely many possible outcomes over monetary amounts. Therefore, $u(x)$ is a standard von Neumann-Morgernstern expected utility. With a slight abuse of notation, we again use u to denote the Bernoulli utility of the von Neumann-Morgernstern representation. Monotonicity of u in this case means $\delta_k \succcurlyeq \delta_m$ when $k \geq m$. With this identification, a bet xEy pays $x = \delta_k$ if E occurs and $y = \delta_m$ if E^c occurs, and we write $u(x) = u(k)$, were the first u is the "expected utility" and the second u is the Bernoulli utility.

Lottery \ Loss	yes	no
win	$x-\pi+W-\ell$	$x-\pi+W$
lose	$x-\pi-\ell$	$x-\pi$

Table 3: Act f_3 - No insurance, Yes ticket

The SSEU of the third act is

$$V(f_3) = (p(L)+\gamma(L))\left[(p(E)+\gamma(E))u(x-\pi+W-\ell)+(1-p(E)-\gamma(E))u(x-\pi-\ell)\right]+ \\ (1-p(L)-\gamma(L))\left[(p(E)+\gamma(E))u(x-\pi+W)+(1-p(E)-\gamma(E))u(x-\pi)\right].$$

Lottery \ Loss	yes	no
win	$x-\pi+W-q$	$x-\pi+W-q$
lose	$x-\pi-q$	$x-\pi-q$

Table 4: Act f_4 - Yes insurance, Yes ticket

Lastly, the individual can purchase both the insurance and the lottery ticket. Table 4 shows the associated payoffs. The SSEU of the fourth act is

$$V(f_4) = (p(E)+\gamma(E))u(x-\pi+W-q)+(1-p(E)-\gamma(E))u(x-\pi-q).$$

The coexistence of gambling and insurance requires $f_4 \succcurlyeq f_1$.

Fact 1. Assume that the utility u is increasing and concave. If winning the lottery has a positive valence such that $p(E)+\gamma(E)\geq 1$, then there is coexistence of gambling and insurance.

Actually, the conditions of Fact 1 imply $f_4 \succcurlyeq f_2 \succcurlyeq f_1$. Moreover, since u is concave, ceteris paribus, the DM always buys insurance, thus $f_4 \succcurlyeq f_3$.

We see that if utility is concave and winning the lottery is a sufficiently attractive event i.e., if $p(E)+\gamma(E) > 1$, an SSEU individual will buy both the insurance policy and the lottery ticket. Note that Fact 1 is true at any given level of wealth x, so that our analysis does not depend on wealth change effects. Our mechanism operates by using the DM's valence to offset risk aversion. A sufficiently high valence creates risk-seeking behavior even for a DM with concave utility function u. The required condition can be stated in terms of what is known as Jensen's gap (for the first part p of the valence representation of ν)

$$u[p(E)x+(1-p(E))y]-[p(E)u(x)+(1-p(E))u(y)].$$

If this gap is less than the quantity $[u(x)-u(y)]\gamma(E)$, then, despite the concavity of u, the DM will choose the bet between x and y over its p-expectation defined as $p(E)x+(1-p(E))y$.

Fact 2. For any bet xEy with $x \succcurlyeq y$, if u is increasing and $p(E) + \gamma(E) \geq 1$, then $V(xEy) \geq u(p(E)x + (1 - p(E))y)$.

The condition $p(E)+\gamma(E) \geq 1$ implies $\gamma(E) \geq 0$. Thus, if u is convex, the preceding inequality is trivially satisfied since the right-hand side is negative and the left-hand side is positive. The interesting case is when u is concave, and yet a "risk averse" DM can prefer a lottery to its expected value.

To link Fact 2 to the coexistence of gambling and insurance, note that if $p(E) + \gamma(E) \geq 1$, then, by Fact 1, $V(f_4) \geq u(x - q)$, but $u(x - q) \geq u(x - q - \pi + W) \geq u(x - q - \pi + p(E)W)$. Therefore, $V(f_4) \geq u(x - q - \pi + p(E)W)$, and $x - q - \pi + p(E)W$ is the expected value of the lottery under p in the presence of insurance.

4.3 Dominated Choice in Strategy-Proof Mechanisms

Recent empirical evidence suggests that some individuals choose dominated (in the sense of first-order stochastic dominance) strategies in strategy-proof mechanisms, such as the Deferred Acceptance (DA) mechanism much studied in the context of school or college choice (Hassidim et al., 2016; Shorrer and Sóvágó, 2023; Dreyfuss et al., 2022).

Flipping and truncation are two examples of choosing a dominated strategy. Under flipping, an individual submits a ranking that reverses the "obvious" order of two alternatives—e.g., ranking a school choice that comes with a fellowship below the same choice without a fellowship. Under truncation, an individual submits a restricted ranking that omits some schools. Both behaviors are inconsistent with standard SEU, because this theory respects first-order stochastic dominance, but are consistent with our SSEU theory.

Consider two schools σ_1 and σ_2. We let $u(x_1) = m_1 > 0$ denote the utility of being matched with school σ_1, $u(x_2) = m_2$ the utility of being matched with school σ_2, and $u(x_0) = 0$ the utility of not being matched. Clearly, being matched with both σ_1 and σ_2 is not possible under the DA. There are four possible rankings which we denote by $\sigma_1 \triangleright \sigma_2$, $\sigma_2 \triangleright \sigma_1$, σ_1, and σ_2. The first two rankings are complete, while the third and fourth are truncations.

Let E_1 be the event of being matched with school σ_1, E_2 the event of being matched with school σ_2, and $E_{\neg 1,2}$ the event of being matched with school σ_2 conditional on not being matched with school σ_1. Submitting the ranking $\sigma_1 \triangleright \sigma_2$ generates the act that yields x_1 if E_1 occurs, x_2 if $E_{\neg 1,2}$ occurs, and x_0 otherwise. The analogous act corresponds to submitting $\sigma_2 \triangleright \sigma_1$. Following Dreyfuss et al. (2022), we assume for simplicity that the probability assessment of being matched to school σ_2 conditional on not being matched to school σ_1 can be written as $p(E_{\neg 1,2}) = (1 - p(E_1))p(E_2)$, and similarly for the event of being matched to σ_1 conditional on not being matched to σ_2.

Submitting a ranking that contains only σ_1 generates the act $x_1E_1x_0$ that yields x_1 if E_1 occurs and x_0 otherwise. Likewise, submitting a ranking that contains only σ_2 generates the act $x_2E_2x_0$

that yields x_1 if E_2 occurs and x_0 otherwise.

Suppose that an individual ranks school σ_1 higher than σ_2, i.e., $m_1 > m_2$. Flipping means that the individual submits the ranking $\sigma_2 \triangleright \sigma_1$. The SSEU of submitting a faithful ranking is $\nu(E_1)m_1 + \nu(E_{\neg 1,2})m_2$, whereas the SSEU of the flipped ranking is $\nu(E_2)m_2 + \nu(E_{\neg 2,1})m_1$. Submitting a flipped ranking is preferred if

$$p(E_1)p(E_2)(m_1 - m_2) < (\gamma(E_{\neg 2,1}) - \gamma(E_1))m_1 - (\gamma(E_{\neg 1,2}) - \gamma(E_2))m_2.$$

The left-hand side of this inequality is always positive, and so the right-hand side has to be sufficiently positive if flipping is to be optimal. For example, suppose that all events in the inequality have valence 0 except for the aversive event $E_{\neg 1,2}$, corresponding to being matched with the less preferred school σ_2 over the more preferred school σ_1. The inequality then becomes

$$p(E_1)p(E_2)(m_1 - m_2) < -m_2\gamma(E_{\neg 1,2}),$$

which will hold if $\gamma(E_{\neg 1,2})$ is sufficiently smaller than 0. Clearly, in the SEU corresponding case to $\gamma = 0$, this inequality is never satisfied.

Truncation by submitting just σ_1 rather than the ranking $\sigma_1 \triangleright \sigma_2$ is preferred if

$$\nu(E_1)m_1 > \nu(E_1)m_1 + \nu(E_{\neg 1,2})m_2,$$

which will hold whenever $\nu(E_{\neg 1,2}) < 0$, i.e., there is aversion to be rejected by the more preferred school.

Our explanation of flipping and truncation behavior in the DA mechanism differs from the explanation given by Dreyfuss et al. (2022). These authors suggest that "an applicant who is likely to get matched with a school will feel a loss when matched with any other school (even a better one); this can create attachment to the high-probability school—an endowment effect for schools." From this idea they conclude that flipping and even truncation might be observed in high loss-averse individuals. Our explanation is not dependent on a (high) likelihood of being matched with a particular school, but on the aversiveness of not being matched with a preferred school.

4.4 Comparative Statics of Valence

Consider two DM's with preferences $\succcurlyeq_1$ and $\succcurlyeq_2$. We show how the DM's can be ranked in terms of the extent to which their preferences depart from the classical. Define, for $i = 1, 2$,

$$P_i = \{s \in S : x\{s\}y \succ_i y \text{ for some } x \succ_i y\},$$

$$N_i = \{s \in S : y \succcurlyeq_i x\{s\}y \text{ for some } x \succcurlyeq_i y\}.$$

Definition 5. Say that DM 1 is **more non-classical** than DM 2 if, for all $f, g, h \in \mathcal{F}$ and $E \subseteq P_1 \cup P_2$,

$$fEh \succcurlyeq_1 gEh \implies fEh \succcurlyeq_2 gEh,$$

$$fEh \succ_1 gEh \implies fEh \succ_2 gEh,$$

and, for every $x, y, z \in X$ with $x \succ y$ and $E \subseteq N_1 \cup N_2$,

$$xEy \sim_1 z \implies xEy \succcurlyeq_2 z.$$

Theorem 3. *The following conditions are equivalent*

(i) DM 1 is more non-classical than DM 2;

(ii) DM 1 and DM 2 admit SSEU representations (u, ν_1) and (u, ν_2) with decompositions $\nu_1 = (p_1, \gamma_1)$ and $\nu_2 = (p_2, \gamma_2)$ such that $u_1 = u_2$, $p_1 = p_2$, and $|\gamma_1| \geq |\gamma_2|$.

In addition, if DM 1 is more non-classical than DM 2, so that $|\gamma_1| \geq |\gamma_2|$, then $b_1 \geq b_2$. To see this, use $|b_1(p^+(s) - p_1^-(s))| \geq |b_2(p^+(s) - p_2^-(s))|$ for all $s \in S$, where $p_1 = p_2 = p$. Choosing a positive s implies $|b_1 p^+(s)| \geq |b_2 p^+(s)|$, or $b_1 \geq b_2$.

In the next example we consider simple parametric specifications of non-classicality.

Example 5 (continuing Examples 1 and 3)**.** Suppose that DM 1 and DM 2 each have one aversive state s_1^* and s_2^*, respectively. Thus, $\gamma_1(s) = b_1(p_1^+(s) - \delta_{s_1^*})$ for some $s_1^* \in S$, and $\gamma_2(s) = b_2(p_2^+(s) - \delta_{s_1^*})$ for some $s_2^* \in S$, Then, if DM 1 is more non-classical than DM 2, then $p_1 = p_2 = p$, $u_1 = u_2$, and $|\gamma_1| \geq |\gamma_2|$. It follows that $s_1^* = s_2^* = s^*$. Using again $|\gamma_1| \geq |\gamma_2|$, we conclude that $|b_1(p(s) - \delta_{s^*})| \geq |b_2(p(s) - \delta_{s^*})|$, which is equivalent to $b_1 \geq b_2$.

Next, consider $\gamma(s) = b(p^+(s) - \frac{1}{|S|})$, so that

$$V(f) = \mathbb{E}_{p^+}[u(f)] - b\left(\sum_{s \in S}(p^+(s) - \frac{1}{|S|})u(f(s))\right).$$

Again, we see that the degree of non-classicality is parameterized by b, where a larger b indicates a larger departure from classicality. △

5 Extension to Signed Choquet Expected Utility

Here, we extend SSEU by relaxing the Independence axiom. Instead, we assume Comonotonic Independence (Schmeidler, 1989) and obtain a Choquet expected utility with respect to a signed capacity. As in the SSEU case, we can decompose the willingness-to-bet on an event E into the sum of a likelihood and a valence: $\nu(E) = \mu(E) + \Gamma(E)$. In the Choquet case, the likelihood component consists of a positive capacity μ with $\mu(S) = 1$, and the valence component is a set function Γ with $\Gamma(S) = \Gamma(\emptyset) = 0$. Finally, all integrals are taken in the sense of Choquet (1953).

Definition 6. We say that (u, ν) is a **signed Choquet subjective expected utility (SCSEU)** representation of $\succcurlyeq$ if $u : X \to \mathbb{R}$ is an affine function and $\nu : 2^S \to \mathbb{R}$, with $\nu(\emptyset) = 0$ and $\nu(S) = 1$, are such that

$$f \succcurlyeq g \iff \int_S u(f) d\nu \geq \int_S u(g) d\nu.$$

Axiom 6 (Comonotonic Independence - CI)**.** If $f, g, h \in \mathcal{F}$ are comonotonic and $\gamma \in (0, 1]$, then $f \succcurlyeq g$ implies $\gamma f + (1 - \gamma)h \succcurlyeq \gamma g + (1 - \gamma)h$.

Theorem 4. *A binary relation $\succcurlyeq$ satisfies axioms WO, CI, A, and IS if and only there exists a non-constant affine function $u : X \to \mathbb{R}$ and a $\nu : 2^S \to \mathbb{R}$ such that (u, ν) is an SCSEU representation of $\succcurlyeq$. Moreover, if (u', ν') is another SCSEU representation of $\succcurlyeq$, then there exist $a, b \in \mathbb{R}$ with $a > 0$ and $u'(x) = au(x) + b$, and $\nu' = \nu$.*

An SCSEU representation can be decomposed as

$$V(f) = \int_S u(f) d\mu + \int_S u(f) d\Gamma$$

where μ is a positive capacity with $\mu(S) = 1$ and Γ is a set function with $\Gamma(S) = \Gamma(\emptyset) = 0$. Indeed, by Cerreia-Vioglio et al. (2012, Proposition 7), any signed capacity of bounded variation ν can be decomposed into $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are positive capacities. Our signed capacity ν is clearly of bounded variation since the state space is finite. Thus, we can write $\nu = a\mu - b\bar{\nu}$, where $a = \nu^+(S)$, $\mu = \nu^+/a$, $b = \nu^-(S)$, and $\bar{\nu} = \nu^-/b$. With these definitions, we get $\mu(S) = \bar{\nu}(S) = 1$, and the normalization $\nu(S) = 1$ implies $a - b = 1$. To obtain the decomposition above, we then set $\Gamma(E) = b(\mu(E) - \bar{\nu}(E))$ for any $E \in 2^S$.

Schneider and Schonger (2019) found experimental evidence for violations of the following consequence of Monotonicity: If $xEz \succ yEz$ for some $x, y, z \in X$ and an event E, then $xEz' \succcurlyeq yEz'$ for all $z' \in X$. We can reproduce this preference pattern using our SCSEU model.

Example 6. Fix payoffs $x, y \in X$ with $x \succ y$ and consider an event E and a capacity ρ with $\rho(E) > 0$. Given $z \in X$ with $y \succcurlyeq z$, we have

$$V(xEz) = \rho(E)u(x) + (1 - \rho(E))u(z) > \rho(E)u(y) + (1 - \rho(E))u(z) = V(yEz).$$

That is, $xEz \succ yEz$. Now suppose that $1 - \rho(E^c) < 0$. Then, given $z' \succcurlyeq x$, we have

$$V(yEz') = \rho(E^c)u(z') + (1 - \rho(E^c))u(y) > \rho(E^c)u(z') + (1 - \rho(E^c))u(x) = V(xEz').$$

That is, $yEz' \succ xEz'$. Notice that if the capacity ρ is a signed measure, then the conditions $\rho(E) > 0$ and $1 - \rho(E^c) < 0$ cannot be simultaneously satisfied. Also, the capacity ρ is not convex, since $\rho(E) + \rho(E^c) > 1$. △

A Appendix

A.1 Extension: BA preferences

Axiom 7 (Risk Independence - RI)**.** If $x, y, z \in X$ and $\gamma \in (0,1]$, $x \succcurlyeq y$ implies $\gamma x + (1-\gamma)z \succcurlyeq \gamma y + (1-\gamma)z$.

Theorem 5. *$\succcurlyeq$ satisfies axioms 1,3,7 and 5 if and only if there exists a non-constant, affine function $u : X \to \mathbb{R}$ and a normalized functional $I : B(u(X)) \to \mathbb{R}$ such that $f \succcurlyeq g$ if and only if $I(u(f)) \geq I(u(g))$.*

Proof. By axiom 1, there is $V : \mathcal{F} \to \mathbb{R}$ that represents $\succcurlyeq$. Since X is mixture set and because $\succcurlyeq$ satisfies axioms 1,3,7 and 5, by Theorem 8 in Herstein and Milnor (1953) there exists $u : X \to \mathbb{R}$ that represents the restriction of $\succcurlyeq$ on X and such that

$$u\,(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y),$$

for every $x, y \in X$ and $\alpha \in [0,1]$. Moreover, u is unique up to positive affine transformations. Further, observe that we can find $\bar{x}, \underline{x} \in X$ such that $u(\bar{x}) = -u(\underline{x}) = 1$. To see this, notice that there exist $\bar{x}, \underline{x} \in X$ such that $\bar{x} \succ \underline{x}$. Indeed, if $x \sim x'$ for every $x, x' \in X$, then it would follow by axiom 5 that $f \sim g$ for every $f, g \in \mathcal{F}$, contradicting axiom 1. Therefore without loss of generality we can take u such that $u(\bar{x}) = 1$ $u(\underline{x}) = -1$.

Define a function $U : \mathcal{F} \to B(u(X))$ as follows: for every $f \in \mathcal{F}$ and $s \in S$

$$U(f)(s) := u(f(s)).$$

Then, U is surjective and U satisfies that $U(f) = U(g) \Rightarrow f \sim g$ by axiom 5. Therefore, U is bijective. Now, define a functional I on $B(u(X))$ by

$$I(\phi) = V\left(U^{-1}(a)\right)$$

for all $\phi \in B(u(X))$. Clearly, it holds that for all $f \in \mathcal{F}$

$$I(U(f)) = I(u \circ f) = V(f).$$

Moreover, $I(k1_S) = k$ for every $x \in u(X)$ with $u(x) = k$. □

B Proofs

B.1 Proof of Proposition 1

The first result provides an intermediate step in the proof of Proposition 1.

Proposition 3. *Assume that* $\succcurlyeq$ *admits the SSEU representation* (u, ν)*. Then there exist* $p^+, p^- \in \Delta^+(S)$*,* $b \geq 0$ *such that*

$$\nu(s) = (1+b)p^+(s) - bp^-(s),$$

for every $s \in S$*, so that for every* $f \in \mathcal{F}$

$$\int_S u(f)d\nu = (1+b)\int_S u(f)dp^+ - b\int_S u(f)dp^-.$$

Moreover, $\succcurlyeq$ *satisfies axiom 4 if and only if* $b = 0$*.*

Proof. Let $a = \nu^+(S)$ and $b = \nu^-(S)$. Clearly $a \geq 0$ and $b \geq 0$ and $\nu(S) = 1$ excludes that $a = b = 0$. If $b = 0$, then $\nu = \nu^+$ and $a = 1$. Moreover, if $a < 1$ then $1 = \nu(S) = a - b$ would imply $b \leq 0$ that contradicts the fact that ν^- is a positive measure, therefore $a \geq 1$. By setting $p^+(s) = \frac{\nu^+(s)}{a}$ and $p^- = \frac{\nu^-(s)}{b}$ (if $b > 0$, otherwise $p^+ = \nu^+ = \nu$) for every $s \in S$, we have that $p^+, p^- \in \Delta^+(S)$. Moreover, we have

$$\nu(s) = \nu^+(s) - \nu^-(s) = \frac{a\nu^+(s)}{a} - \frac{b\nu^-(s)}{b} = ap^+(s) - bp^-(s) \text{ for every } s \in S.$$

Now let

$$N = \{s \in S : x\{s\}y \succ y \text{ for some } y \succ x\},$$

and observe that $s \in N$ if and only if $\nu(s) < 0$. If $\succcurlyeq$ satisfies monotonicity, then $N = \emptyset$, and since $b = \nu^-(S) = \nu^-(S \cap N) = \nu^-(\emptyset) = 0$. Hence $b = 0$ as desired. If $b = 0$, then $\succcurlyeq$ satisfies monotonicity. □

Proof of Proposition 1. By defining $\nu(s) = p(s) + \gamma(s)$, a Valence representation can be rewritten as $V(f) = \int_S u(f)d(p+\gamma) = \int_S u(f)d\nu$. The set function ν is such that, for any $E, F \subseteq S$ with $E \cap F = \emptyset$, $\nu(E \cup F) = p(E \cup F) + \gamma(E \cup F) = p(E) + p(F) + \gamma(E) + \gamma(F) = \nu(E) + \nu(F)$, where the second equality follows from p being a probability and property 1 of γ in Definition 1. Lastly, point 2 in Definition 1 implies that $\nu(S) = p(S) + \gamma(S) = 1 + 0 = 1$, so that $\nu \in \Delta(S)$.

For the opposite implication, suppose $\succcurlyeq$ has a SSEU representation (u, ν). By Proposition 3, there is $b \geq 0$, and $p^+, p^- \in \Delta^+(S)$ such that $\nu(s) = (1+b)p^+(s) - bp^-(s)$. The Valence representation follows by defining $p(s) = p^+(s)$ and $\gamma(s) = b(p^+(s) - p^-(s))$. □

B.2 Proof of Theorem 1

Several steps of this proof are standard, but we report them for completeness. We denote with B the set of functions from S to $\mathbb{R}$, and given $K \subseteq \mathbb{R}$ we denote with $B(K)$ the set of functions from S to K.

Since $\mathcal{F}$ is mixture set and because $\succcurlyeq$ satisfies axioms 1-3, by Theorem 8 in Herstein and Milnor (1953) there exists $V : \mathcal{F} \to \mathbb{R}$ that represents $\succcurlyeq$ such that

$$V(\alpha f + (1-\alpha)g) = \alpha V(f) + (1-\alpha)V(g),$$

for every $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$. Moreover, V is unique up to positive affine transformations. Further, observe that we can find $\bar{x}, \underline{x} \in X$ such that $V(\bar{x}) = -V(\underline{x}) = 1$. To see this, notice that there exist $\bar{x}, \underline{x} \in X$ such that $\bar{x} \succ \underline{x}$. Indeed, if $x \sim x'$ for every $x, x' \in X$, then it would follow by axiom 5 that $f \sim g$ for every $f, g \in \mathcal{F}$, contradicting axiom 1. Therefore without loss of generality we can take V such that $V(\bar{x}) = 1$ $V(\underline{x}) = -1$.

Define $u : X \to \mathbb{R}$ by $u(x) = V(x)$ for every $x \in X$. Let $K := u(X)$. Then K is convex by the affinity of V and without loss of generality satisfies $[-1,1] \subseteq K$ by the previous paragraph.

Define a function $U : \mathcal{F} \to B(K)$ as follows: for every $f \in \mathcal{F}$ and $s \in S$

$$U(f)(s) := u(f(s)).$$

Then, U is surjective since $K = u(X)$ and U satisfies that $U(f) = U(g) \Rightarrow f \sim g$ by axiom 5. Therefore, U is bijective. Furthermore, the affinity of V implies that for all α and $f, g \in \mathcal{F}$

$$U(\alpha f + (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g).$$

Now, define a functional I on $B(K)$ by

$$I(\phi) = V\left(U^{-1}(a)\right)$$

for all $\phi \in B(K)$. Clearly, it holds that for all $f \in \mathcal{F}$

$$I(U(f)) = I(u \circ f) = V(f).$$

Moreover, $I(x1_S) = x$ for every $x \in K$.

Now let $\phi, \psi \in B(K)$ and let $\alpha \in [0,1]$. Also, let $f, g \in \mathcal{F}$ be such that $U(f) = a$ and $U(g) = b$. Since U is surjective, such f and g exist. Then, we obtain

$$\begin{aligned}
I(\alpha a + (1-\alpha)b) &= V\left(U^{-1}(\alpha a + (1-\alpha)b)\right) \\
&= V\left(U^{-1}(\alpha U(f) + (1-\alpha)U(g))\right) \\
&= V\left(U^{-1}(U(\alpha f + (1-\alpha)g))\right) \\
&= V(\alpha f + (1-\alpha)g) \\
&= \alpha V(f) + (1-\alpha)V(g) \\
&= \alpha I(a) + (1-\alpha)I(b).
\end{aligned}$$

It follows that I is positively homogeneous, i.e. for every $\alpha > 0$ and $\phi \in B(K)$ such that $\alpha\phi \in B(K)$ it holds $I(\alpha\phi) = \alpha I(\phi)$ (see for example). Moreveor, I is additive, that is for every $\phi, \psi \in B(K)$ such that $\phi + \psi \in B(K)$ it holds $I(\phi + \psi) = I(\phi) + I(\psi)$. To see this point, observe that by positive homogeneity we obtain $I(\phi + \psi) = I\left(2\left(\frac{\phi}{2} + \frac{\psi}{2}\right)\right) = 2I\left(\frac{\phi}{2} + \frac{\psi}{2}\right) = 2\left(\frac{I(\phi)}{2} + \frac{I(\psi)}{2}\right) = I(\phi) + I(\psi)$.

We can now extend I to B as follows. For every $\phi \in B$ there exists $\psi \in B(K)$ and $\alpha \geq 1$ such that $\psi = \alpha\phi$. We can therefore define $\bar{I} : B \to \mathbb{R}$ by

$$\bar{I}(\phi) = \alpha I(\psi).$$

Observe that it is now immediate to satisfies $\bar{I}$ satisfies for every $\alpha \in \mathbb{R}$ and $\phi, \psi \in B$

1. $\bar{I}(\phi + \psi) = \bar{I}(\phi) + \bar{I}(\psi)$;

2. $\bar{I}(\alpha\phi) = \alpha\bar{I}(\phi)$.

Now by the Riesz representation theorem (e.g., see Theorem 6.45 in Axler (1997)) there exists a (unique) $\nu : S \to \mathbb{R}$ such that for every $\phi \in B$

$$\bar{I}(\phi) = \int \phi d\nu = \sum_{s \in S} \phi(s)\nu(s).$$

We can therefore conclude that

$$f \succcurlyeq g \iff V(f) \geq V(g) \iff I(u(f)) \geq I(u(g)) \iff \int u(f)d\nu \geq \int u(g)d\nu.$$

Note that we can assume that $\nu(S) = \sum_{s \in S} \nu(s) = 1$. Indeed, it must be that $\sum_{s \in S} \nu(s) = \nu(S) \neq 0$ since $I(1_S) = 1$. Further, if $\sum_{s \in S} \nu(s) = \nu(S) \neq 1$, then letting let $\tilde{\nu}(s) = \frac{\nu(s)}{\nu(S)}$ and $\tilde{u} = \nu(S)u$, so that

$$V(f) = I(u(f)) = \int u(f)\nu = \int \tilde{u}(f)\tilde{\nu},$$

so that the claim is satisfied.

Finally, if there exists $\nu' \in \Delta(S)$ and an affine function $u' : X \to \mathbb{R}$ such that (u', ν') represents $\succcurlyeq$, then because V is unique up to affine transformations there exists $a > 0$ and $b \in \mathbb{R}$ such that

$$\int u'(f)d\nu' = a \int u(f)d\nu + b$$

for every $f \in \mathcal{F}$ from which we obtain

$$u'(x) = au(x) + b,$$

for every $x \in X$. We can therefore conclude that $\nu' = \nu$ as desired.

B.3 Proof of Theorem 2

It is easy to see that $E \subseteq S$ is $\succcurlyeq$-null iff $\nu(E) = p(E) + \gamma(E) = 0$. This immediately implies the characterization of $\succcurlyeq$-completely null events, by additivity of ν.

Next, using the definitions of p^+ and p^- from Proposition 3, we see that for every $s \in S$ we have $\nu(s) = 0$ iff $p(s) = p^+(s) = 0$ and $p^-(s) = 0$. (For the "only if" direction, notice that by definition $\nu(s) = 0$ implies $\nu^+(s) = 0 = \nu^-(s)$, which immediately implies $p^+(s) = 0$ and $p^-(s) = 0$.) Analogously, $\nu(s) > 0$ iff $p(s) = p^+(s) > 0$ and $p^-(s) = 0$, and $\nu(s) < 0$ iff $p(s) = p^+(s) = 0$ and $p^-(s) > 0$. If E is $\succcurlyeq$-classically null, it cannot contain any state s such that $x\{s\}y \succ y$ for some $x \succ y$ (which would imply $\nu(s) > 0$), hence $p(E) = 0$. Conversely, if $p(E) = 0$, E cannot clearly contain any event F such that $p(F) > 0$, and so it is $\succcurlyeq$-classically null.

B.4 Proof of Fact 1

According to SSEU, the difference between $V(f_2)$ and $V(f_1)$ is

$$u(x-q) - \nu(L)u(x-\ell) - (1-\nu(L))u(x).$$

Since $0 \leq \nu(L) \leq 1$, Jensen's inequality yields

$$V(f_2) - V(f_1) = u(x-q) - \nu(L)u(x-\ell) - (1-\nu(L))u(x) \geq$$
$$u(x-q) - u(x-\nu(L)\ell) = 0.$$

Thus, the DM buys the insurance policy. The SSEU difference between f_4 and f_2 is

$$V(f_4) - V(f_2) = \nu(E)u(x-q-\pi+W) + (1-\nu(E))u(x-q-\pi) - u(x-q).$$

By assumption, $\nu(E) = p(E)+\gamma(E) = 1+\epsilon$ for some $\epsilon \geq 0$. Also, $u(x-q-\pi+W)-u(x-q) \geq 0$ since u is increasing and $W \geq \pi$. Similarly, $u(x-q-\pi+W) - u(x-q-\pi) \geq 0$. Therefore

$$u(x-q-\pi+W) + \epsilon(u(x-q-\pi+W) - u(x-q-\pi)) - u(x-q) \geq 0,$$

because $W \geq \pi$, and thus $V(f_4) \succcurlyeq V(f_2)$. We conclude that $V(f_4) \succcurlyeq V(f_2) \succcurlyeq V(f_1)$.

B.5 Proof of Fact 2

If $p(E) + \gamma(E) \geq 1$, then $\gamma(E) \geq 1 - p(E)$. Since $u(x) \geq u(y)$, we have $\gamma(E)[u(x) - u(y)] \geq (1-p(E))[u(x) - u(y)]$. Adding and subtracting $p(E)u(x)$ on the right-hand side implies

$$\gamma(E)[u(x) - u(y)] \geq u(x) - [p(E)u(x) + (1-p(E))u(y)].$$

Since u is increasing, we can write $u(x) \geq u(p(E)x + (1-p(E))y)$,and thus

$$\gamma(E)[u(x) - u(y)] \geq u(p(E)x + (1-p(E))y) - [p(E)u(x) + (1-p(E))u(y)],$$

as required.

B.6 Proof of Theorem 3

$(i) \implies (ii)$: First observe that 1 and 2 admit SSEU representations given by $(u, \nu_1), (u, \nu_2)$ such that $u_1 = u_2$. To see this, observe that since 1 is more classical than 2, taking $f = x, g = y$ it follows that

$$x \succcurlyeq_1 y \iff xEh \succcurlyeq_1 yEh \iff xEh \succcurlyeq_2 yEh \iff x \succcurlyeq_2 y,$$

for all $x, y \in X$. Further observe that $P_1 = P_2 := P$. To see this, take $s \in P_1$. Then it must be that $x\{s\}y \succ_1 y$. Since 1 is more classical than 2, it follows that letting $x\{s\}y \succ_2 y$, which implies $s \in P_2$, so that $P_1 \subseteq P_2$. One can show in the same way that $P_2 \subseteq P_1$. It follows that $N_1 = N_2 := N$.

Now define $\succsim'_i$, $i = 1, 2$ on X^P as follows

$$f \succsim'_i g \iff \text{there exists } h \in \mathcal{F} \text{ such that } fPh \succsim_i gPh,$$

for every $f, g \in X^P$. Since 1 is more classical than 2, we obtain that

$$f \succsim'_1 g \iff f \succsim'_2 g,$$

it follows that

$$p_1(s) = \frac{\nu_1(s)}{\nu_1(P)} = \frac{\nu_2(s)}{\nu_2(P)} = p_2(s) \text{ for every } s \in P.$$

Finally, given any $s \in N$, choose x, y, z such that $x\{s\}y \sim_1 z$.[9] Since 1 is more classical than 2 it follows that

$$\nu_1(s) = \frac{u(y) - u(z)}{u(x)} \leq \nu_2(s),$$

which implies that $\gamma_1(s) \leq \gamma_2(s)$ whenever $s \in N$. When $s \in P$, $\gamma_1(s) = b_1 p(s) \geq b_2 p(s) = \gamma_2(s)$. We can therefore conclude that $|\gamma_1| \geq |\gamma_2|$.

$(ii) \implies (i)$: Assume that $p_1 = p_2 = p$. Take $f, g, h \in \mathcal{F}$ and $E \subseteq P_1 \cup P_2$. We have

$$fEh \succsim_1 gEh \iff \int_S u(f)dp_1 \geq \int_S u(g)dp \iff fEh \succsim_2 gEh.$$

Now take x, y, z with $u(x) > u(y)$ and $E \subseteq N$ such that

$$xEy \sim_1 z,$$

which is equivalent to $\gamma_1(E) = \frac{u(z)}{u(x)-u(y)}$. Since $E \subseteq N$, it follows that $\gamma_2(E) \geq \gamma_1(E)$, which is equivalent to

$$xEy \succsim_2 z,$$

as desired.

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[9]Such triple x, y, z always exist. Let y be such that $u(y) = 0$ and choose x, z so that $\nu_1(s) = \frac{u(z)}{u(x)}$.

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