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# Reevaluating the Shapley Value: A New Justification and New Extensions

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Abstract

Inspired by the bargaining procedure of Shapley (1953), we introduce a novel procedure in which the marginal contribution of the player joining a coalition is split between that player and the members of the coalition being joined. Surprisingly, this more general procedure leads to the Shapley value. This finding offers a new understanding of and motivation for the Shapley value. Our procedure also leads to a non-cooperative game (analogous to Hart and Mas-Colell, 1996) in which expected payoffs in the unique stationary subgame-perfect equilibrium equal the Shapley value. These results rely on a random-ordering assumption. Moving away from that case leads to a new generalization of the Shapley value, one in which the weights on players depend endogenously on marginal contributions. We extend our analysis to NTU games, where it generalizes the consistent solution of Maschler and Owen (1989, 1992).

Keywords: Shapley value, $n$-person bargaining, generalized bargaining procedures, value division, non-cooperative implementation, weighted Shapley value, NTU games, consistent solutions.

## 1. Introduction

The Shapley value (Shapley, 1953) results from an axiomatic approach to the allocation of value in a cooperative game. It has an elegant procedural implementation:

> Consider all $n!$ possible orderings of $n$ players. Assume each ordering is equally likely. In each ordering, give all the marginal value created by the new player joining the coalition to that player. The expected value is the Shapley value.

In what we will call the Shapley procedure, the "negotiation" between the new player and the existing players involves an extreme division of value. The player who joins a

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coalition receives the full benefit, i.e., the full marginal contribution to the coalition. While the Shapley value is defined by its axioms, not the associated procedure, the procedure summarizes a potential negotiation that leads to the Shapley value and thus provides valuable intuition. We strengthen this intuition by exploring a procedure (and an associated non-cooperative game) in which the gains to the negotiation between a coalition and an outside player are more evenly balanced among the parties.

In our alternative procedure, the player joining the existing coalition shares the gains with the members of the coalition. The gains are shared in the proportion $(\alpha,\ 1-\alpha)$. When $\alpha = 1$, we get the original Shapley procedure. When $\alpha < 1$, the existing members benefit when the additional player joins, and that should make them more willing to accept the joiner. We see this as a more appealing procedure. We still need to describe how the $1-\alpha$ gain is shared among members of the existing coalition. In particular, we do not advocate equal division. Our proposed internal division, which is based on marginal contributions, better reflects bargaining power. In one special case —when the joining player has the largest marginal contribution—some portion of the $1-\alpha$ gain is allocated back to the joining player. In that sense, we say that the existing members have "priority" on the $1-\alpha$ gain.

Surprisingly, our alternative procedure yields the Shapley value for all $0 \leq \alpha \leq 1$. This result provides additional support for and understanding of the Shapley value. It shows what is essential to Shapley's procedure is the random ordering of how players join existing coalitions, not the allocation of all the marginal contribution to the player joining.

Dividing up the joiner's contribution among the members of an existing coalition implies that each member's payoff depends on the marginal contributions of players other than itself. But the Shapley value payoff to a player depends only on that player's marginal contributions. (This fact characterizes the Shapley value; see Young, 1985.) In our procedure—and in the non-cooperative game that follows it—the division of value among the members of the existing coalition depends on the relative marginal contributions of all the different members. While each player obtains less than its marginal contribution when joining, the extra amount it collects when other players join its coalitions exactly offsets the difference and thus the Shapley value emerges.

A procedure such as ours is a way of understanding a cooperative solution concept, but it is not a non-cooperative implementation of the concept. We provide a fully non-cooperative game that mimics our procedure and builds on the non-cooperative game in Hart and Mas-Colell (1996). Our non-cooperative game also suggests a new family of generalized Shapley values, different from but related to the well-known weighted Shapley value (Shapley, 1953).

We go on to extend our procedure to non-transferable utility (NTU) games. Now, unlike the transferable utility (TU) case, the solution depends on the $(\alpha,\ 1-\alpha)$ split. For two-player games, the solution coincides with the Nash (1950) bargaining solution for $0 \leq \alpha \leq 1$.

Little work seems to have been done on modifications to Shapley's procedure. In contrast, there is a long-standing program to find non-cooperative foundations for cooperative solution concepts. Gul (1989, 1999) initiated this program in the case of the Shapley value, with further contributions from Winter (1994), Hart and Mas-Colell (1996), Hart and Levy (1999), Serrano (2005; 2019), and McQuillin and Sugden, (2016).

In Gul (1989), players effectively sell their negotiation rights to other players, and the result is a series of pairwise negotiations. Players are randomly matched. One side makes a proposal to the other. If the receiver accepts, the proposer becomes the residual claimant for the group. That player makes or rejects offers for the combined group going forward and keeps all the value created net of payments. (If the other side rejects, there is some discounting and another random matching arises.) Thus, the Gul model consists of a series of two-player negotiations. As the discount rate converges to 1, any *efficient* result converges to the Shapley value. Gul (1999) provides certain convexity assumptions that ensure there is an efficient equilibrium; see also McQuillin and Sugden (2016).

In Winter (1994), players are randomly picked in turn to make a demand. If there is a set of compatible demands, those players depart the game with their demands being met. Here, too, the demands—more precisely, the expected values of the demands—equal the Shapley value provided that the game is convex.[1]

We are interested in an approach that works for all games, not just convex ones. Hart and Mas-Colell (1996) provide the first such model. A player $i$ is randomly chosen to make an offer to the other players in the coalition. In effect, player $i$ presents an ultimatum to the others. The procedure is easiest to follow in the case where, in the event of no deal, the player making the proposal is eliminated from the game and gets nothing. In that case, player $i$ makes an ultimatum in which it is given its marginal contribution and all the other players are offered their Shapley values in the game without player $i$. In the unique stationary subgame-perfect equilibrium of this game, the expected payoffs equal the Shapley values. If the game is repeated with a probability approaching $1$ in the event of no agreement, then the random payoffs converge to the Shapley values. Rather than being randomly chosen, if players bid to

[1] A game is convex when the marginal contribution of a player to a coalition is increasing as a coalition gains more members.

make a proposal, the equilibrium of the game also results in the players getting their Shapley values; see Pérez-Castrillo and Wettstein (2001) and Mutuswami, Pérez-Castrillo, and Wettstein (2004).

Hart and Mas-Colell (1996) provide a generalization in which the player at risk of exclusion is not necessarily the person making the offer. This is the general avenue we pursue, although we consider a different game tree. In our game, a player is chosen at random to be at risk of exclusion in the event of no agreement. With chance $\alpha$, that player is given the opportunity to make a proposal to the rest of the players. With chance $1-\alpha$, one of the rest of the players are given priority to make a proposal. The key feature of our model is that in dividing up the $1-\alpha$ probability, the chance a player is chosen to make a proposal depends on the marginal contributions.

We think it is natural to give each player a share of the total that is connected to its marginal contribution in the relevant coalition, as in Hart and Mas-Colell (1989). But giving each player its marginal contribution doesn't work since it might be infeasible or inefficient. In our approach, players with greater marginal contributions are more likely to be chosen. In this way, the division of the value depends on the players' marginal contributions. In spite of this more natural and complicated interdependency, our game still yields the Shapley value. More precisely, it has a unique stationary subgame-perfect equilibrium and the associated expected payoffs are the Shapley values. Unlike Hart and Mas-Colell, the payoffs equal the Shapley value even when the person at risk of being excluded is different from the person making the offer.

We begin in Section 2 with a motivating example of our procedure in a simple three-person game and we show that the procedure leads to the Shapley value. Section 3 presents our general framework and the proof that our procedure yields the Shapley value for all $0 \leq \alpha \leq 1$. It is important to emphasize that our procedure was chosen because we think it is economically intuitive, and not for the result it gets. This section ends with a defense of our procedure, which includes a uniqueness result (proved in the Appendix).

In Section 4, the procedure is used to build an associated non-cooperative game. We prove the game has a unique stationary subgame-perfect equilibrium independent of $\alpha$ and the expected payoffs in this equilibrium are the same as in our bargaining procedure, namely, the Shapley values. Section 5 presents an intuitive justification for our specific game.

The result that equilibrium payoffs are independent of $\alpha$ depends on the assumption that all players have an equal chance of being the one subject to potential exclusion. An alternative assumption, explored in Section 6, is that in the event of a breakdown, some subgroups—for example, those that create more value—are more likely to form than others. Thus, a natural generalization is to consider the case where the chance a player is subject to exclusion depends on that player's marginal contribution. (For example, the player with the lowest marginal contribution might be the most natural to exclude.) The result of this generalization is similar to the weighted Shapley value (Shapley, 1953; Kalai and Samet, 1987), but differs in that the weights are endogenous to the game. In these circumstances, the solution depends on $\alpha$.

While the focus of the paper is on transferable-utility (TU) games, we provide an extension to non-transferable-utility (NTU) games in Section 7. We offer a brief conclusion in Section 8.

## 2. A Motivating Example

Three players $A$, $B$, and $C$ have the potential to create value by working together. No player can create any value acting alone. The efficient solution is a three-way partnership:

$$v(A) = v(B) = v(C) = 0,$$

$$v(AB) = 20, v(AC) = 18, v(BC) = 12,$$

$$v(ABC) = 24.$$

Our procedure describes a method for resolving the negotiation among $A$, $B$, and $C$ over how to divide the 24 units of value. Similar to the Shapley procedure, we have players joining existing coalitions in a random order. Unlike Shapley, the marginal contribution created by the joining player is divided up $(\alpha, 1 - \alpha)$ with $\alpha$ going to the joining member. If $\alpha = 1$, the two procedures coincide.

We believe our generalization offers an important insight into how negotiations might proceed, since it allows for a more even split of gains between a joining player and an existing coalition. In particular, one might imagine that $\alpha = 1/2$. The challenge is that we have to decide how to allocate the $(1 - \alpha)$ share among the members of the existing coalition. Simply dividing the amount equally would be unsatisfactory. It would mean that players who bring nothing, that is, dummy players, get positive payoffs. Our approach is to allocate the $(1 - \alpha)$ share in relation to the marginal contributions of the players inside the existing coalition. Each player has a claim on the allocation up to their marginal contribution. Players share equally in those claims.

For the first two players who join, it is immediate that their expected payoffs are independent of $\alpha$. Consider the situation where $A$ and $B$ are the first two to join though not necessarily in that order. Half the time $A$ will join $B$ and half the time $B$ will join $A$. The player joining second has a marginal contribution of $20$, either way. Thus, on average, $A$ and $B$ will each get

$$20 \times [\frac{1}{2}\alpha + \frac{1}{2}(1-\alpha)] = 20/2 = 10.$$

Half the time $A$ gets share $\alpha$ and half the time share $1-\alpha$. The expected payoff is independent of $\alpha$ for $A$ (and also for $B$). An identical argument can be made for the coalitions $\{AC\}$ and $\{BC\}$. The interesting case concerns the expected payoffs that arise when a third player joins an existing coalition of two players.

For example, if $C$ were to join $\{AB\}$, there is a marginal contribution of 4 to be split. Player $C$ gets $\alpha 4$, while $A$ and $B$ each get $0.5(1-\alpha)4$. Both $A$ and $B$ have marginal contributions above 4 and thus both have a claim on the entire 4 units of value. The procedure splits their joint share evenly.

The situation is different when $B$ joins $\{AC\}$. Now there is a marginal contribution of $6$ to be split. Player $B$ gets $\alpha 6$, while $A$ and $C$ each get $0.5(1-\alpha)4$, and then $A$ gets all the additional $(1-\alpha)2$. Both $A$ and $C$ have a claim on the first $4$ units of marginal contribution and so they split this amount evenly. Player $A$'s marginal contribution is $12$ and so $A$ is the only inside player to have a claim on the part of $B$'s marginal contribution between 4 and $6$. For this reason, $A$ doesn't have to share this part with $C$, whose marginal contribution is only 4.

There is one further complexity that arises when the player joining has the largest marginal contribution. Consider the situation when $A$ joins $\{BC\}$. Now there is a marginal contribution of $12$ to be split. Player $A$ gets $\alpha 12$, while $B$ and $C$ each get $0.5(1-\alpha)4$, and then $B$ gets all of the next $(1-\alpha)2$ of value. Both $B$ and $C$ have a claim on the first 4 units of marginal contribution and so their share of the 4 is evenly split. Player $B$'s marginal contribution is $6$ and so $B$ is the only inside player to make a claim on the part of $A$'s marginal contribution between $4$ and $6$. For this reason, $B$ doesn't have to share this part with $C$. Note, though, that none of the inside players have a claim on the part of $A$'s marginal contribution between $6$ and $12$. Our procedure returns this share to $A$. The total to $A$ is thus $\alpha 12 + (1-\alpha)6 = \alpha 6 + 6$.

We have repeated all these payoffs in Table 1 below. For ease in reading the table, we have put the payoff to the player joining the coalition in bold.

| | Payoff to $A$ | Payoff to $B$ | Payoff to $C$ | Total |
|---|---|---|---|---|
| $C$ joins $\{AB\}$ | $2(1-\alpha)$ | $2(1-\alpha)$ | $4\boldsymbol{\alpha}$ | $4$ |
| $B$ joins $\{AC\}$ | $4(1-\alpha)$ | $6\boldsymbol{\alpha}$ | $2(1-\alpha)$ | $6$ |
| $A$ joins $\{BC\}$ | $6\boldsymbol{\alpha}+6$ | $4(1-\alpha)$ | $2(1-\alpha)$ | $12$ |
| Average | $12/3$ | $6/3$ | $4/3$ | |

**Table 1**

Because all three negotiations are equally likely, the expected value to each player is the average payoff. The key observation is that each party's expected payment is independent of $\alpha$.

Consider, for example, the expected payoff to $A$ for the last step in the procedure, the one in which the coalition grows from two to three members. Under the Shapley procedure, player $A$ would get its full marginal contribution of $12$ with chance $1/3$ (when it is the one joining $\{BC\}$). Under our more general procedure, it only receives $\alpha 6+6$ in that case, but it also receives $6(1-\alpha)$ across the two cases where $C$ or $B$ joins a two-person coalition with $A$. Thus, on average, player $A$ still receives $12/3$, the same expected payoff as with the Shapley procedure. Parallel arguments apply to $B$ and $C$. Each player obtains less than its marginal contribution when it joins a coalition but this is exactly balanced by the extra amount it collects when other players join the coalitions to which it belongs.

The conventional approach focuses on $\alpha=1$: All the marginal contribution goes to the party joining the coalition. Another interesting case is $\alpha=1/2$: The marginal contribution is split evenly, with half going to the player joining the coalition and the other half going to the coalition. Under $\alpha=1/3$, the marginal contribution is split according to relative size. There is one player outside and two players inside. The player joining gets an equal per-capita share. (Indeed, it is even possible to let $\alpha$ vary with the size of the coalition—say, $\alpha=1/3$ when a player joins a two-party coalition and $\alpha=1/4$ when a player joins a three-party coalition.) Lastly, there is the opposite to the Shapley approach, namely, $\alpha=0$: All the marginal contribution goes to the (inside) coalition. Our result shows that all these cases yield the same expected payoffs to the players.

As we next show, our example generalizes to all TU games. This establishes that beyond the traditional approach there is an intuitive and general family of negotiation procedures that yields the Shapley value.

## 3. The $\alpha$-Procedure

We now define the general procedure for TU games and show that the expected payoffs to the players are independent of $\alpha$. (This immediately implies that our general procedure yields the Shapley value.) Later, we provide a non-cooperative implementation of our game, one in which the equilibrium payoff to each player coincides with the payoff under our procedure.

We fix a player set $N$. For each subset $S \subseteq N$, there is a real number $v(S)$, to be interpreted as the value created by the players in $S$. We choose the normalization $v(\emptyset) = 0$. We also assume $v(\{i\}) \geq 0$ for all $i$ and that $v$ is super-additive, that is, if $S \cap T = \emptyset$ then $v(S \cup T) \geq v(S) + v(T)$.

Let the set $S$ without player $i \in S$ be denoted by $S\backslash i$. Denote the marginal contribution of player $i \in S$ to set $S$ by $m_i(S) = v(S) - v(S\backslash i)$. Note that $m_i(S) \geq v(\{i\}) \geq 0$ where the first inequality follows from super-additivity and the second from the non-negativity assumption.

Let the number of players in coalition $S$ be denoted by $|S|$. We adopt the following naming convention: For each $S$, order the players according to their marginal contributions to $S$, so that $m_1(S) \leq m_2(S) \leq \ldots \leq m_{|S|}(S)$. In the event of equality of two marginal contributions, choose an arbitrary order. Note this ordering is not constant across $S$. In addition, following Gul (1989), we create a dummy player $0$, that is, a player $0$ for whom $m_0(S) = 0$ for all $S$. This is done to simplify the summation notation that follows.

In our procedure, like the Shapley procedure, players join existing coalitions in a random order. Unlike the Shapley procedure, the gains are split $(\alpha, 1-\alpha)$ between the joiner and the existing coalition members. We require $0 \leq \alpha \leq 1$ to ensure that, under our procedure, all players gain from the negotiation and are therefore willing to accept the outcome. For each $i \in S$, when player $i$ joins $S\backslash i$, player $i$ gets $\alpha m_i$.

It remains to specify the division of the remaining $(1-\alpha)m_i$ among the $|S|-1$ players in $S\backslash i$. The procedure divides $(1-\alpha)m_i$ evenly among the players who have a claim to that value. That is, for each $j \in S\backslash i$, player $j$ gets an additional amount

$$(1-\alpha)\sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S|-k}\,. \tag{1}$$

When $i = |S|$, there is one additional allocation. In this case, even though $i$ is not part of the coalition being joined, it also gets

$$(1-\alpha)(m_{|S|} - m_{|S|-1})\,. \tag{2}$$

We call the procedure just defined the $\alpha$-procedure.

Each player inside the sub-coalition shares equally in the sub-coalition's negotiation gains up to that player's marginal contribution. Thus player $1$ (defined as the player with the lowest marginal contribution) only gets $\frac{(1-\alpha)m_1}{|S|-1}$ no matter who the outside player is. Because player $1$ then drops out of subsequent sharing, the remaining gains are divided up among one less player.

Player $2$ gets $\frac{(1-\alpha)m_1}{|S|-1} + \frac{(1-\alpha)(m_2-m_1)}{|S|-2}$. This is the same share of $m_1$ as player $1$ gets. Then, as player $1$ has no further claim, player 2 gets the same share of $m_2 - m_1$ as the $|S|-2$ players whose marginal contributions exceed $m_1$.

Continuing in this fashion, player $j$ shares each stage $k$ in the summation in Equation (1) equally with all the players $i \geq k$, since their marginal contributions are all at least as large as $k$'s marginal contribution. The last stage to be divided is $m_i - m_{i-1}$, since at that point all of the $(1-\alpha)m_i$ has been allocated.

When the joining player is $i = |S|$, the summation in Equation (1) stops at $|S|-1$. Since no player inside the sub-coalition has a claim on anything exceeding that player's marginal contribution, there are no players $j \in S\backslash|S|$ who have a claim on $m_{|S|} - m_{|S|-1}$. That is why the amount $(1-\alpha)(m_{|S|} - m_{|S|-1|})$ reverts back to player $|S|$, as seen in Equation (2). We say that the coalition $S\backslash i$ has "priority" in terms of getting the $(1-\alpha)m_i$; these players get the entire amount except when $i = |S|$, since then there is a last segment of marginal contribution on which no player in $S\backslash i$ has a claim.

Note that the $\alpha$-procedure ensures that no player gets more than its marginal contribution. In particular, a dummy player, that is, player with zero marginal contribution, gets zero.

**Theorem 1: In the $\alpha$-procedure, the expected payoffs to all players are independent of $\alpha$ and equal to the Shapley value.**

**Proof:** Consider a coalition $S$ and player $j \in S$. Player $j$'s expected payoff is made up of two components. There is the amount player $j$ receives across all $|S|-1$ cases where a player $i \in S\backslash j$ joins the coalition. There is also the amount player $j$ receives when joining $S\backslash j$. For $j < |S|$, player $j$'s total expected value $\pi^j$ from the procedure is

$$\pi^j = \frac{\alpha m_j}{|S|} + \frac{(1-\alpha)}{|S|} \sum_{i \neq j}^{|S|} \sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} \tag{3}$$

$$= \frac{\alpha m_j}{|S|} + \frac{(1-\alpha)}{|S|} \sum_{k=1}^{j} (m_k - m_{k-1}) \tag{4}$$

$$= \frac{\alpha m_j}{|S|} + \frac{(1-\alpha) m_j}{|S|} = \frac{m_j}{|S|}. \tag{5}$$

Equation (4) follows since the $\dfrac{m_k - m_{k-1}}{|S| - k}$ terms appear $|S| - k$ times in Equation (3), once for every value of $i \geq k$ excluding $i \neq j$.

For player $j = |S|$, the summation in Equation (3) is still $j$'s average payoff when $j$ is part of the coalition being joined. However, the maximum term in the first summation is $i = |S| - 1$. Thus the summation for player $|S|$ in Equation (4) stops at $|S| - 1$, not at $|S|$, so that

$$\frac{1-\alpha}{|S|} \sum_{k=1}^{|S|-1} (m_k - m_{k-1}) = \frac{(1-\alpha) m_{|S|-1}}{|S|}. \tag{6}$$

In addition, player $|S|$ receives

$$\alpha m_{|S|} + (1-\alpha)(m_{|S|} - m_{|S|-1|}), \tag{7}$$

when joining from outside. The player with the largest marginal contribution only has to share the marginal contribution of the player with the second-highest marginal contribution and keeps all of the last increment of its marginal contribution. Thus player $|S|$'s expected value from the procedure is

$$\pi^{|S|} = \frac{(1-\alpha)m_{|S|-1} + \alpha m_{|S|} + (1-\alpha)(m_{|S|} - m_{|S|-1})}{|S|} = \frac{m_{|S|}}{|S|}. \tag{8}$$

The negotiation payoffs to all players are independent of $\alpha$ for each coalition and thus must also be independent of $\alpha$ when summing up across coalitions. The expected payoffs to all players must therefore equal the Shapley value since the procedure coincides with the Shapley procedure when $\alpha = 1$, and the expected payoffs are independent of $\alpha$. ■

This result may be somewhat surprising. A player's Shapley value is a function only of that player's marginal contributions. In the $\alpha$-procedure, when player $i$ joins $S\backslash i$, the payoffs to players $j \in S\backslash i$ depend on a constellation of marginal contributions. However, Theorem 1 shows that in the expected value calculation, the different marginal contributions cancel out in exactly the right way so that each player's expected payoff depends only on that player's marginal contributions.

We want to emphasize that the $\alpha$-procedure was not picked with the goal of having marginal contributions cancel in order to arrive at the Shapley value. In defining the $\alpha$-procedure, we faced a new and previously unaddressed problem of allocating the sub-coalition's payoff among its members when $\alpha < 1$. (When $\alpha = 1$, the sub-coalition being joined gets nothing so the division question is moot.) We think the $\alpha$-procedure is the most natural one in that it treats players equally up to their claims. Based on Theorem 1, the Shapley value can be thought of as arising from a procedure or negotiation process that allows for *any* $\alpha$ split between the joiner and the group being joined. The key assumption that leads to the Shapley value is that the order of joining is random, not that the gains all go to the person joining the coalition.

We can ask to what extent is the $\alpha$-procedure unique in terms of leading to the Shapley value.[2] Consider all procedures where the player joining the sub-coalition receives $\alpha$ of the total value and the players being joined have priority on the other $1-\alpha$ share. The question is whether, within this class, there are other procedures for dividing up the $1-\alpha$ share that also lead to the Shapley value.

In the Appendix, we show that our $\alpha$-procedure is the unique bargaining procedure, within this general class, that yields the Shapley value if the procedure satisfies three axioms. The axioms require that: (i) claims are limited to marginal contributions; (ii) stronger parties (as defined by their marginal contributions) get more; and (iii) players get less when the other players in the sub-coalition are stronger. Thus, not only do we think our division rule for the $1-\alpha$ share is intuitive, we establish that among all possible procedures satisfying our axioms, the $\alpha$-procedure uniquely leads to the Shapley value.

### 4. A Non-Cooperative Implementation

We next develop a non-cooperative game that mimics our procedure and implements the Shapley value. This connection adds further credence to our $\alpha$-procedure. There is, however, one important difference between the procedure and the game. The game naturally starts with the full set of players present and considers negotiations that might lead to the elimination of one of the players. For this reason, it will help in building intuition—and our game tree—to develop a version of our original procedure that runs in reverse. Instead of building the coalitions up by adding one player at a time (in a random order), we can think of starting with the full set of players and potentially reducing the set of players one at a time (in a random order). We emphasize that the two procedures are fully equivalent: For each random construction of the coalition, there is an equivalent deconstruction, as we now explain.

To determine the division of $v(N)$, some player $i \in N$ is chosen at random. In this reverse version of the $\alpha$-procedure, player $i$ "negotiates" with the others about its share. The procedure assigns player $i$ a share $\alpha$ of its marginal contribution while the others split the $(1-\alpha)$ of player $i$'s marginal contribution in the same fashion as in the $\alpha$-procedure.[3] Player $i$ then exits the game. To determine the split of the remaining

[2] We thank Sergiu Hart for directing us to consider this question.

[3] As in the original procedure, if the player has the maximal marginal contribution, it gets an additional $(1-\alpha)(m_{|S|}-m_{|S|-1|})$.

$v(N\backslash i)$ among the remaining players, the deconstruction process repeats: A player $j$ from the set $S = N\backslash i$ is chosen at random and there is an $(\alpha, 1 - \alpha)$ division of $j$'s marginal contribution just as before. The process continues with $S\backslash j$ until there is only one player remaining. This is the same recursion as in our original procedure. The payoffs are the same when run forwards or backwards and therefore equal the Shapley value.

To calculate the expected payoffs in the reverse procedure, one has to run through the $N!$ possible unwindings of the full coalition. While the unwinding procedure is less familiar than the standard Shapley building-up procedure, the advantage of the reverse procedure is that it naturally leads to a non-cooperative game, as we now present.

Our non-cooperative game is similar in spirit to the game in Hart and Mas-Colell (1996) but formulated with the reverse $\alpha$-procedure in mind. To develop intuition, the game starts with all the players present simultaneously—they are all in a room together—and they consider what would happen if they fail to reach a joint agreement. There are two relevant players in each stage of the negotiation. There is a player who makes a proposal to all the other players. And there is a player, possibly but not generally the same, who is at risk of being eliminated if the proposal is rejected. (In the case when $\alpha = 1$, the proposer is always the at-risk player.)

Following Hart and Mas-Colell (1996), our non-cooperative game consists of (potentially) infinitely many rounds of negotiation.[4] In each round there is a set $S \subseteq N$ of active players, starting with $S = N$ in the first round. One of the players $i \in S$ is chosen to be at risk of exclusion in the event of a breakdown; this selection is done according to a given probability distribution $\sigma = (\sigma_i)_{i \in S}$. We allow the probabilities $\sigma_i$ to depend on $S$ and $v|S$ (the restriction of $v$ to $S$). This is more general than in Hart and Mas-Colell, where these probabilities depend only on $S$. We want to allow for the possibility that some players are more likely to be at risk of exclusion than others, such as those players with low marginal contributions.

Once the player at risk of exclusion has been identified, a player is chosen to make a proposal; this is done according to a given probability distribution $\tau = (\tau_j)_{j \in S}$. These probabilities also may depend on $S$ and $v|S$, and the identity of the player at risk. Thus players with high marginal contributions may be more likely to make offers than players

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[4] The order of play is different from Hart and Mas-Colell. They start with the proposer and then determine the at-risk player, while we start with the at-risk player and then select the proposer. All proofs carry over from one case to the other.

with lower marginal contributions. Let $\tau_{j|i}$ be the conditional probability that $j$ is chosen to make a proposal, given that $i$ is at risk.

Knowing the player at risk of exclusion, the proposer makes a proposal that specifies payoffs to all players. The members of $S$ are then asked in some pre-specified order whether or not they accept the proposal. If they all accept, the game ends with these payoffs. Otherwise, the proposal is rejected and the game moves to the next round. With probability $\rho$ the set of active players remains unchanged ("repeat"), while with probability $1-\rho$ the player subject to elimination drops out and gets a final payoff of $0$ ("breakdown"). For simplicity, we assume that $\rho$ is independent of $\sigma$ and $\tau$, although our results do not depend on this. More precisely, the set of active players in the next round is $S$ with probability $\rho$, and $S\backslash i$ (where $i$ is the player at risk) with probability $1-\rho$.

We now need to specify the $\sigma_i$ and $\tau_{j|i}$. In the main part of their paper, Hart and Mas-Colell assume $\sigma_i = 1/|S|$ for all $i$ and $\tau_{j|i} = 1$ for $j = i$ and $0$ otherwise. In this case, each player in $S$ is equally likely to be chosen to be at risk, and that same player is the one to make a proposal. (The case $\tau_{i|i} = 1$ is akin to $\alpha = 1$ in our procedure.) These assumptions lead to the Shapley value in the following sense. For all $\rho$, the expected payoffs to the players in the unique stationary subgame-perfect equilibrium are their Shapley values. Moreover, the random payoffs all converge to the Shapley values as $\rho \to 1$. As in Hart and Mas-Colell (1996), we consider throughout the stationary subgame-perfect equilibria of the game. For simplicity, we will simply call these the equilibria of the game.

It is helpful to describe the equilibrium of the Hart and Mas-Colell game with $\rho = 0$. The game starts with the full set of players $N$ and an individual is randomly chosen to make an offer (and risk being eliminated if the offer is rejected). That player will ask for its marginal contribution. If it asks for any more, it will be turned down; if it asks for any less, it leaves money on the table. The chosen player offers all the other players their Shapley values in $N\backslash i$ since otherwise someone in that coalition would rather reject the proposal.

Now consider the Shapley procedure run in reverse, i.e., our $\alpha$-procedure when $\alpha = 1$. A player is chosen at random from $N$ and the procedure gives that player its marginal contribution. The procedure continues with the coalition $N\backslash i$. A player is chosen at random from $N\backslash i$ and the procedure gives that player its marginal contribution. The procedure continues until no one is left. The set $N$ is randomly deconstructed and in each step a player obtains its marginal contribution as its payoff. The expected payoffs to the players in $N$ and those left at any stage are their Shapley values at that stage. Thus, the expected payoffs to all players left in $N\backslash i$ are their Shapley values in $N\backslash i$. We can think of the procedure as giving the chosen player its marginal contribution and giving the others players expected values equal to their Shapley values in the coalition without that player, just as in the (unique) equilibrium of the game.

Note the similarity between these two analyses. This observation motivates our choice of $\tau_{j|i}$ for the case of general $\alpha$, not just $\alpha = 1$. We maintain the assumption that $\sigma_i = 1/|S|$.

Let player $i$ be chosen to be at risk. The proposer is then selected as follows. With chance $\alpha$, player $i$ is also chosen to make a proposal to the other players in $S$. With chance $1 - \alpha$, the not-at-risk players are given priority in terms of making a proposal. The proposer is then chosen according to the steps below:

(1) a number is selected at random from the interval $[0,\ m_i(S)]$;

(2) any player $j \neq i$ whose marginal contribution $m_j(S)$ is above the number selected is in the eligible pool;

(3) all players in the eligible pool are selected with equal probability;

(4) if no player is in the eligible pool, player $i$ makes the proposal.[5]

The underlying idea is that assigning proposer probabilities is akin to assigning payoffs. Just as in our $\alpha$-procedure, we want the game to divide payoffs (probabilities) equally among the members of a coalition who have a claim to that value. Thus the payoff $m_i(S)$ is divided up into $i$ intervals—$m_1(S) - m_0(S), m_2(S) - m_1(S), \ldots, m_i(S) - m_{i-1}(S)$—and the probability of being chosen in each of these intervals is evenly divided up among all members whose marginal contributions exceed the upper bound of that interval.

Combining the chance of making an offer when subject to exclusion with the chance of making an offer in the $(1 - \alpha)$ case, the total chance player $j \in S$ will make an offer given player $i$ is at risk is:

[5] This step is only relevant when $i = |S|$.

$$\tau_{j|i} = \begin{cases} \alpha & \text{for } j = i < |S|; \\ \alpha + \frac{(1-\alpha)}{m_{|S|}} [m_{|S|} - m_{|S|-1}] & \text{for } j = i = |S|; \\ \frac{(1-\alpha)}{m_i} \sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} & \text{for } j \neq i\,. \end{cases} \tag{9}$$

For $\alpha = 1$, this game essentially replicates the basic game in Hart and Mas-Colell (1996).[6] When $0 \leq \alpha < 1$, our game is novel. In particular, the chance of making a proposal depends on marginal contributions and not just $S$. We provide further intuition for our specific choice of $\tau_{j|i}$ in Section 5.

The $\tau_{j|i}$ are closely related to the payoffs under our $\alpha$-procedure. When the randomly chosen player is $i$, the expected payoff to each player $j \in S$ under our procedure is $\tau_{j|i} \times m_i$. We show that when $\sigma_i = 1/|S|$ and $\tau_{j|i}$ is as in Equation (9), the expected payoff to each player in the unique equilibrium is identical to that in the $\alpha$-procedure and therefore equals the Shapley value.

Let $\phi^k(S)$ be player $k$'s expected payoff for $k \in S$ and set $\phi^k(S) = 0$ for $k \notin S$.

**Theorem 2: Fix a TU game. If $\sigma_i = 1/|S|$ for all $i \in S$, $S \subseteq N$, and $\tau_{j|i}$ is as defined in Equation (9), then, for $0 \leq \alpha \leq 1$, in any stationary subgame-perfect equilibrium:**

$$\phi^k(S) = \frac{1}{|S|}[m_k + \sum_{i \in S} \phi^k(S \backslash i)]. \tag{10}$$

**Proof:** It follows from Hart and Mas-Colell (1996, Proposition 9) there is a unique equilibrium and the expected payoffs satisfy

$$\phi^k(S) = \sum_{i \in S} \sigma_i [\phi^k(S \backslash i) + \tau_{k|i} m_i]. \tag{11}$$

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6 Hart and Mas-Colell also consider more general specifications such as when the proposer is never excluded in the event of a breakdown, and all other players have an equal chance of being excluded: $\sigma_i = 1/|S|, \tau_{i|i} = 0, \tau_{j|i} = 1/(|S| - 1)$. This case yields equal payoffs to all players. Such alternative specifications carry over to our game, but, as in Hart and Mas-Colell, they lead to expected payoffs different from the Shapley value.

Thus the theorem will hold provided

$$\sum_{i \in S} \tau_{k|i} m_i = m_k \,. \tag{12}$$

Returning to the definition of $\tau_{ji}$, we have

$$\tau_{j|i} m_i = \begin{cases} \alpha m_i & \text{for } j = i < |S|; \\ \alpha m_{|S|} + (1-\alpha)\, [m_{|S|} - m_{|S|-1}] & \text{for } j = i = |S|; \\ (1-\alpha) \sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} & \text{for } j \neq i \,. \end{cases} \tag{13}$$

Note that all three cases are perfectly symmetric between $i$ and $j$. Thus

$$\tau_{j|i} m_i = \tau_{i|j} m_j, \tag{14}$$

and, therefore,

$$\sum_{i \in S} \tau_{k|i} m_i = \sum_{i \in S} \tau_{i|k} m_k = m_k \sum_{i \in S} \tau_{i|k} = m_k. \tag{15}$$

The final equality in Equation (15) follows from the fact that in the four-step procedure that defines $\tau_{i|k}$, some player is always chosen to make the proposal when $k$ is at risk; thus the probabilities must sum to $1$. ▪

**Corollary 1:** Under the assumptions of Theorem $2$, the expected payoffs in the stationary subgame-perfect equilibrium coincide with the Shapley value if and only if

$$\sum_{i \in S} \tau_{k|i} m_i = m_k \,. \tag{16}$$

Proof**:** The if part is clear. Under the conditions in the corollary, the equation for $\phi^k(S)$ in Theorem 2 is the Shapley recursion formula. The if result also follows from the fact that $\phi^k(S)$ is not a function of $\alpha$ (even though the $\tau_{j|i}$ are). We know from Hart and Mas-Colell (1996) that $\phi^k(S)$ coincides with the Shapley value when $\alpha = 1$. Since our $\phi^k(S)$ do not depend on $\alpha$, they must equal the Shapley value for any $\alpha$.

Turning to the only if part, we note that the Shapley value allocation is strongly monotonic (Young, 1985), which implies that the allocation to player $k$ depends only on the marginal contribution of that player. Since $\phi^k(S)$ cannot change with $m_j$ for $j \neq k$, the equality in Equation (16) is also a necessary condition for $\phi^k(S)$ to be the Shapley value for $k$. ■

By itself, this corollary does not prove that our $\tau_{k|i}$ are the unique weights that satisfy Equation (16) and thus lead to the Shapley value when $\sigma_i = 1/|S|$. However, Theorem A.1 in the Appendix shows that if the weights satisfying three natural axioms, the $\tau_{k|i}$'s are indeed unique.

For $\sigma_i = 1/|S|$, our results extend Hart and Mas-Colell (1996) to the case where the chance of making an offer can depend on the marginal contributions of all the players. In Hart and Mas-Colell, where the probabilities depend only on $S$, the expected payoffs in the unique equilibrium coincide with the Shapley value if and only if $\sigma_k(S) = 1/|S|$, $\tau_{k|k}(S) = 1$, and $\tau_{k|i}(S) = 0$ when $i \neq k$, for all $k \in S$. As they explain (p. 375): "[T]o obtain the Shapley value one needs, first, that only proposers (but not responders) may drop out; and second, that the probabilities … of dropping out should be equalized across the players."

In our framework, once the probability of dropping is equalized across players, we obtain the Shapley value even when players other than proposers drop out and when the probabilities of making a proposal are not equalized. The reason is that we allow $\tau_{k|i}(S)$ to depend on $S$ and $v|S$ (the restriction of $v$ to $S$). In particular, the probabilities depend on the marginal contributions with respect to $S$.

Note that the $\phi^k(S)$ are independent of $\rho$. The potential breakdown probability $\rho$ does not change the expected payoffs to players in the game. As $\rho \to 1$, it follows that all offers made will converge to the Shapley values. Let $\phi^k_{ji}(S)$ be player $k$'s expected payoff when player $i$ is at risk of elimination and player $j$ is chosen to make the offer.

**Corollary 2:** Under the assumptions of Theorem $2$, in the unique stationary subgame-perfect equilibrium, $\phi_{ji}^{k}(S) \to \phi^{k}(S)$ as $\rho \to 1$.

**Proof:** As in Hart and Mas-Colell (1996), player $j$ has to offer each player $k$ at least its expected value from non-agreement in order for $k$ to agree. There is no reason for $j$ to offer more. Thus

$$\phi_{ij}^{k}(S) = \rho\phi^{k}(S) + (1-\rho)\phi^{k}(S\backslash i) \qquad \text{for} \quad k \neq j\,. \tag{17}$$

That leaves player $j$ with

$$\phi_{ij}^{j} = \rho\phi^{j}(S) + (1-\rho)[\phi^{j}(S\backslash i)) + m_j(S)]. \tag{18}$$

As $\rho \to 1$, $\phi_{ij}^{k}(S) \to \phi^{k}(S)$ for all $k \in S$. ▪

Letting $\rho \to 1$ leads all payoffs to converge to their expected values. When the risk of elimination is near zero, there is little gain in making the offer and little concern about being eliminated. Thus, all payoffs converge to their expectations, which are the Shapley values.

## 5. Justifying the Non-Cooperative Game

Making a proposal is always advantageous. The player making the proposal gets what it would have achieved when someone else makes the proposal plus $(1-\rho)m_k(S)$, the at-risk part of the marginal contribution of the player subject to elimination. As we explain the intuition for our choice of $\tau_{j|i}$, it will help to focus on the case $\rho = 0$, so that non-agreement leads to elimination and the proposer gets the full marginal contribution of the at-risk player.

There are two sources of power in our model: $\alpha$ and the $\tau_{j|i}$. The choice of $\alpha$ is meant to reflect the split between the "outside" and "inside" players. The power of the "outside" player, the one at risk of elimination, is its chance of making an offer.

The novel aspect of our game is the split among the members of the inside coalition as determined by the $\tau_{j|i}$. It is natural that a player's negotiation power is connected to its marginal contribution. But why should the probabilities be allocated according to the specific formula in Equation (9)?

We illustrate the intuition behind our choice of the $\tau_{j|i}$ using an example with three airlines. We pick this example because the underlying economics help make clear why the $\tau_{j|i}$ naturally vary with the marginal contributions as in Equation (9).

Suppose airline A requires a runway of length $1$, airlines B and C each require a runway of length $2$. The cost per unit length is $1$. The airlines can save money by sharing a common runway:

$$v(A) = v(B) = v(C) = 0;\ v(AB) = v(AC) = 1,\ v(BC) = 2;\ v(ABC) = 3.$$

Consider first the $\tau_{j|i}$ when airline $A$ is at risk of elimination. Whoever makes the proposal will capture airline $A$'s marginal contribution of $1$ (since $\rho = 0$). These are the savings attributed to airline $A$'s joint use of the first leg of the runway. Airlines $B$ and $C$ both require use of the first leg and should share equally in its cost. The purpose of the negotiation is to allow airline $A$ to share in the use of the first leg. Since airlines $B$ and $C$ are symmetric with regard to the first leg, they should have equal power in the negotiation with airline $A$. Hence, conditional on being in the $1-\alpha$ case, the probability of making the proposal should be $\tau_{C|A} = \tau_{B|A} = 1/2$ .

Consider next the case where airline $B$ is at risk of being eliminated, and where airlines $A$ and $C$ have priority in terms of making the proposal. Whoever makes the proposal will capture airline $B$'s marginal contribution of $2$. These are the savings attributed to airline $B$'s joint use of the initial two legs of the runway. It is helpful to break these savings into two parts, one corresponding to each leg. On the first leg, airlines $A$ and $C$ are symmetric and so should share equally in the savings. For the second leg, only airline $C$ uses this leg and thus should not expect airline $A$ to contribute to its cost. Thus airline $A$ should not have any claim to the second leg of $B$'s marginal contribution. Instead of having a $50$ percent chance of making a proposal that nets $B$'s marginal contribution of $2$, airline $A$ should have a $50$ percent chance of making a proposal only on the first part of $B$'s contribution. It follows that airline $A$'s chance of making the proposal is given by $\tau_{A|B} = 1/2 \times 1/2 = 1/4$. Airline $C$'s chance of making the proposal is given by $\tau_{C|B} = 1/2 \times 1/2 + 1/2 \times 1 = 3/4$, since airline $C$ is the only player with a claim to the savings on the second leg. The same logic says that when airline $C$ is excluded: $\tau_{B|C} = 3/4$ and $\tau_{A|C} = 1/4$. These six $\tau_{j|i}$ are exactly the probabilities provided in Equation (9).

The one case not illustrated in the airline example arises when the maximal marginal contribution is not a tie ($m_{|S|} > m_{|S|-1}$). In this event, some of the $1 - \alpha$ probability reverts back to player $|S|$ when it is at risk of elimination. The reasoning is parallel to the airline example. None of the participants "inside" the coalition have a claim on this last interval of marginal product, $m_{|S|} - m_{|S|-1}$. Thus it should revert to the player who has the unchallenged maximal marginal contribution.

## 6. Generalized Shapley Values

In the weighted Shapley value (Shapley, 1953; Kalai and Samet, 1987), weights in a negotiation are assigned based on the players' identities. Here, we introduce a notion of weights that determine the $\sigma_i$ probabilities. Unlike in Shapley, our weighting is endogenous: It depends on the marginal contributions.

In Theorem 2, we assumed $\sigma_i(S) = 1/|S|$ for all $i \in S$ and $S \subseteq N$. More generally, some players might be at greater risk of elimination than others.[7] For example, $\sigma_i > 0$ might only apply to the player(s) with the lowest marginal contribution—if a coalition breaks down, it does so in the least-expensive way. We allow $\sigma_i(S)$ to depend on $v|S$ (though we suppress this dependence in our notation). While any dependence is possible, we think the parametrized family below is an interesting case. Assume $m_j(S) > 0$ for some $j \in S$.[8] The chance that player $i$ is at risk of elimination is

$$\sigma_i(S) = \frac{m_i(S)^\lambda}{\sum_{j \in S} m_j(S)^\lambda}. \tag{19}$$

In particular, if $\lambda = 0$, then $\sigma_i(S) = 1/|S|$, which leads to the Shapley value. The case $\lambda = -1$ makes player $i$'s chance of elimination inversely proportional to its marginal contribution. In the limiting case of $\lambda \to -\infty$, only the player(s) with the lowest marginal contribution have a positive chance of elimination. The lower the value of $\lambda$, the greater the risk of elimination for the players with low marginal contributions. In all cases other than $\lambda = 0$, the payoffs depend on $\alpha$.

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[7] This idea is similar in spirit to Owen (1977) and Hart and Kurz (1983) where some coalitions are more likely to form than others. Here, we attach probabilities to which coalitions form in the event of a negotiation breakdown.

[8] The case where all players have no marginal contribution to $S$ is uninteresting since there is no value to divide up in the negotiation.

A key distinguishing feature of these weights is that they are anonymous and endogenous. In particular, for any two-player game, $m_1(S) = m_2(S)$, and so both players have an equal chance of elimination. Thus, unlike the weighted Shapley value (Shapley, 1953) which would generally favor either player $1$ or player $2$, here the weighting function only comes into play when the marginal contributions are different, which can happen only in games with three or more players.

We illustrate the $\lambda \to -\infty$ weighting rule by returning to our airline example. Airline $A$ has the lowest marginal contribution to $ABC$. Thus, airline $A$ would be the one at risk of exclusion in $\{ABC\}$: $\sigma_A = 1$, $\sigma_B = \sigma_C = 0$. In the sub-coalition $\{BC\}$, the marginal contributions are equal and so $B$ and $C$ have an equal chance of being at risk of exclusion: $\sigma_B = \sigma_C = 1/2$.

This leads to two equally likely exclusion orderings: $A$ then $B$, or $A$ then $C$. The resulting equilibrium expected payoffs to the three airlines are:

$$\phi^A = \alpha,\ \phi^B = \phi^C = 1 + (1 - \alpha)/2.$$

Unlike our previous solution, the expected payoffs depend on $\alpha$. Airline $A$ might do better or worse than its payoff of $2/3$ in the Shapley value. The fact that airline $A$ is the player subject to exclusion does not in itself put $A$ at a disadvantage. Power comes from making the proposal. In the Shapley setup, we can think of $A$ as having a two-thirds chance of making a proposal—it is $1/3$ in the initial round and another $2/3 \times 1/2 = 1/3$ in the second round. In the weighted case, the chance the at-risk player makes the proposal is $\alpha$. If $\alpha = 1$, then it is advantageous to be the player at risk. If $\alpha = 1/2$, so that the inside and outside player bargaining powers are equal, airline $A$ does slightly worse than under the Shapley value. In our example, airlines $B$ and $C$ will form the maximally profitable partnership in the event of a breakdown. Thus airline $A$'s only chance to make an offer is in the first round. Its payoff is $\alpha$, which is the chance the at-risk player makes the offer.

Our non-cooperative game suggests a different extension of the Shapley value. Under Shapley's weighted approach, different players come with different power. Even in a two-person negotiation, the gains are not split evenly. In our case, we preserve the idea that there will be an even split in any two-person negotiation (since marginal contributions are always equal). Instead, we imagine that the player subject to potential elimination might not be chosen at random. In particular, the chance of being subject to elimination might depend on marginal contributions along with other aspects of the characteristic function.

## 7. Extension to NTU Games

We show how our $\alpha$-procedure can be used to define an NTU game. For $\alpha = 1$, this problem has been addressed in Hart and Mas-Colell (1996), who provide a procedure that yields the consistent Shapley value (Maschler and Owen, 1989; 1992). When $0 \le \alpha < 1$, a new procedure is required, just as in the TU case, since we need a rule to share the $1 - \alpha$ portion of value. We do so by defining the NTU marginal contribution of a player to a set $S$, which leads to our generalized procedure.

Given an NTU game $(N, V)$, we assume that the feasible sets $V(S)$ satisfy the standard conditions on the characteristic function; see, in particular, conditions (A.1)–(A.3) in Hart and Mas-Colell (1996). Let $\partial V(S)$ denote the boundary of the feasible set for $S$. For convenience, we perform two normalizations. We set $\partial V(i) = \{0\}$. We also scale the utilities for all players so that the maximum feasible utility level of each player $i$ in $V(N)$ is 1.

We begin with the case where $\partial V(N)$ is a hyperplane (therefore, the unit simplex under our scaling) and then show how to extend our analysis to the general convex case as in Maschler and Owen (1989; 1992). Let $\Psi$ denote the vector of payoffs from the procedure. By our normalization $\Psi(i) = 0$ for one-player games.

Assume inductively that we have a solution for coalitions of size up to $|N| - 1$ (for any characteristic function). Fix a game $(N, V)$. We derive the solution for the set $N$. The marginal contributions associated with $N$ are

$$d^i(N) := \max\{c^i : (c^i, \Psi(N \backslash i)) \in V(N)\}. \tag{20}$$

This the maximal possible payoff to $i$ given that the other players obtain their payoffs in the game without $i$.[9]

With our set of marginal contributions, and following our earlier numbering convention, we index the players in order of increasing marginal contribution. The inductive step in our NTU procedure is obtained by adapting our earlier TU game.

[9] This definition is different from that in Hart and Mas-Colell (1996) in two ways. First, the marginal contributions defined in Equation (20) are independent of the order of player arrival. Second, outside of a hyperplane game, the solution to the subgame $\Psi(N \backslash i)$ need not equal the average marginal contribution of each player in that game. In Hart and Mas-Colell, the marginal contributions are defined inductively based on a specific ordering of player arrivals. For a hyperplane game—one in which $V(S)$, for $S \subseteq N$, is a half space—our $d^i(N)$ equal the average value of Hart and Mas-Colell's marginal contributions across all $(N-1)!$ possible orderings that build to $N$ in which player $i$ is the last to join.

From the set $N$ we randomly select a player to be at risk. Given player $i$ is at risk, we assign the probability $\tau_{j|i}$ of player $j$ being the proposer exactly as in Equation (9), simply substituting the $d^i(S)$ for the $m_i(S)$. In that way, the parameter $\alpha$ enters into the procedure.

The procedure (contingent on the random selection of $i$ and $j$) assigns everyone their value in $\Psi(N \backslash i)$, with the proposer $j$ receiving an additional $d^i(N)$.[10] Because $\partial V(N)$ is a hyperplane (normalized to the unit simplex), it is always efficient and feasible to assign $d^i(N)$ to the player making the proposal. The payoffs $\Psi(N)$ are the expected values where each player has an equal chance of being at risk,

$$\Psi(N) = \frac{1}{|N|} \sum_{i=1}^{|N|} [\Psi(N \backslash i) + \sum_{j=1}^{|N|} \tau_{ji} d^i(N \backslash i) e^j]. \tag{21}$$

and where $e^j$ is the $j$th unit vector. Again, because $\partial V(N)$ is a hyperplane, this expected value is both efficient and feasible.

Moreover, the same argument as in Equation (15) shows:

$$\frac{1}{|N|} \sum_{i=1}^{|N|} \tau_{ji} d^i(N \backslash i) = d^j(N \backslash i). \tag{22}$$

Thus

$$\Psi(N) = d(N) + \frac{1}{|N|} \sum_{i=1}^{|N|} \Psi(N \backslash i), \tag{23}$$

where $d(N)$ is the vector of the $d^j(N)$. We can see that this procedural solution is the NTU analog to the Shapley recursion relationship.

However, this is only the solution for the case where $\partial V(N)$ is a hyperplane. To find the procedural solution for general $V(N)$, we look for a fixed point as in Maschler and Owen (1992). Fix a point $p$ on the unit simplex. Consider the ray from the origin

[10] As before, we extend $\Psi(N \backslash i)$ so that player $i$—who is not part of $N \backslash i$—receives $0$ in $\Psi(N \backslash i)$.

through $p$. This ray will intersect $\partial V(N)$ at some point $q$. Let hyperplane $H(q)$ be tangent to $\partial V(N)$ at $q$. Normalize $H(q)$ so that it is the unit simplex. Apply the same scaling to $V(N)$. Consider the game when the scaled $V(N)$ is extended to $H(q)$. Here, the boundary is a hyperplane, so we can apply the preceding solution $\Psi(N)$. This is a continuous mapping from the unit simplex to itself—from $p$ to $q$ to $\Psi(N)$—and thus has a fixed point. The fixed point is defined as the consistent solution to the original $V(N)$, and this completes the procedure. The intuition for selecting the fixed point is similar to the axiom of Independence of Irrelevant Alternatives: $\Psi(N)$ is a solution for a larger set that includes $V(N)$ and it remains feasible in the smaller set $V(N)$, so it should be the solution in the smaller set.

Observe that the inductive step has two parts. We start with $|N|$ players and randomly break the set into $|N| - 1$ insiders and one "at-risk" player. We apply the procedure to a game with $|N| - 1$ players, and divide up the at-risk player's contribution to obtain the solution to a game with $|N|$ players. This first step is done when the boundary for $V(N)$ is a hyperplane. We then use the solution to all such games to find a fixed point for general $V(N)$. This is similar to the way the Nash bargaining solution is constructed.

We offer some remarks on our NTU procedure. First, if the game is TU, the procedure leads to the same result as our $\alpha$-procedure defined in Equations (1)–(2). Next, for two-person games, our NTU procedure leads to the Nash (1950) bargaining solution for all values of $\alpha$. When the boundary of the bargaining set is a line, the NTU procedure selects the midpoint: $\Psi(N) = 1/2[(\alpha, 1 - \alpha) + (1 - \alpha, \alpha)] = (1/2, 1/2)$. As in the Nash bargaining solution (Nash, 1950), the NTU procedure for convex sets selects the boundary point which is the midpoint of the tangent line at that boundary point.

For $\alpha = 1$, our procedure leads to the same consistent solution(s) as in Hart and Mas-Colell.[11] Any consistent solution is based on the solution to a hyperplane game and our procedures align in hyperplane games when $\alpha = 1$.

## 8. Conclusion

This article introduces a new bargaining procedure, one that is more general and that allows for greater balance between the person joining a coalition and the members of

[11] Hart and Mas-Colell allow for a penalty in the case of disagreement that we set to $0$. The solutions coincide when the penalty is $0$.

that coalition. The gains are shared $(\alpha,\ 1-\alpha)$ between the outsider and the insiders, rather than $(1,0)$.

We showed, perhaps surprisingly, that the $\alpha$-procedure leads to the Shapley value for all values of $0 \leq \alpha \leq 1$. We think this leads to a better understanding of what underpins the Shapley value: The essential feature is the random ordering of how players join existing coalitions, not the allocation of all the marginal contribution to the player joining.

The $\alpha$-procedure can be grounded in a non-cooperative game which is a generalization of that presented in Hart and Mas-Colell (1996). A new feature of this generalized game is that the expected payoffs are the Shapley values even when the proposer and the player at risk of exclusion are not the same.

The $\alpha$-procedure and the game suggest a generalization of the Shapley value, different from the weighted Shapley value. Specifically, the weights are endogenous—they depend on the marginal contributions—rather than being exogenously determined by the players' identities. The motivation is that in the event of a breakdown, some coalitions are more likely to form than others. With just two players, there are no other coalitions to form and so the division remains even for all two-person TU games.

Lastly, we showed how the $\alpha$-procedure can be used to define a consistent solution for NTU games following the approaches in Maschler and Owen (1992) and Hart and Mas-Colell (1996). In the special case of two-person games, the $\alpha$-procedure leads to the Nash (1950) bargaining solution.

## 9. References


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## 10. Appendix

Before providing our uniqueness result in Theorem A.1, we present the relevant axioms and the associated notation. Consider the negotiations among group $S$, as $S$ goes from size $|S|-1$ to $|S|$. As before, order the players by their marginal contributions, $m_i(S)$. For each player $i \in S$, when $i$ joins $S \backslash i$, it gets $\alpha m_i$. The general question is how to divide up the remaining $(1-\alpha)m_i$. We find it simpler to think of the negotiation as players in $S \backslash i$ dividing up $m_i$ and then multiply those allocations by $(1-\alpha)$.[12]

Our first axiom is based on the idea that players only compete for claims up to their marginal value.

**Axiom 1:** In the competition for $m_i$, the claim of member $j \in S \backslash i$ is limited by its marginal contribution $m_j$.

All members of $S \backslash i$ have a claim on amounts up to $m_1$, members $2,3,...,|S|$ of $S \backslash i$ have a claim on amounts up to $m_2$, …, and members $i+1, i+2, ..., |S|$ of $S \backslash i$ have a claim on amounts up to $m_i$.[13] Consequently, we can think of the negotiation over $m_i$ as a series of claims on $m_1$, $m_2 - m_1$, …, $m_i - m_{i-1}$, which are the increments of marginal contribution.

We next introduce the weights $w_{jk}(S \backslash i)$ that determine each player $j$'s share of the $k$th interval of $m_i$. We define $w_{jk}(S \backslash i) = 0$ for $k > i$ since $m_i$ is divided into $i$ intervals.

Summing over the $i$ intervals, player $j$ receives in total

$$(1-\alpha)\sum_{k=1}^{i} w_{jk}(S \backslash i)(m_k - m_{k-1}) \,. \qquad \text{(A1)}$$

Since they are weights,

$$\sum_{j \in S} w_{jk}(S \backslash i) = 1 \text{ for any } 1 \leq k \leq i \,. \qquad \text{(A2)}$$

From Axiom 1, we can set $w_{jk}(S \backslash i) = 0$ for $j < k$. Thus player $j$ receives

[12] When $i = |S|$ and $m_{|S|} > m_{|S|-1}$ some of $(1-\alpha)m_i$ reverts back to player $i$.

[13] Note that $i \notin S \backslash i$, so that the final set starts with $i+1$.

$$(1-\alpha)\sum_{k=1}^{Min(i,j)} w_{jk}(S\backslash i)(m_k - m_{k-1}). \qquad \text{(A3)}$$

With one exception, $w_{ik}(S\backslash i) = 0$, since player $i$ is on the outside. For $i = |S|$, no player in $S\backslash|S|$ has a claim on $(m_{|S|} - m_{|S|-1})$ and so that interval reverts back to player $|S|$. By Axiom 1, $w_{j|S|}(S\backslash|S|) = 0$ for $j < |S|$ and thus $w_{|S||S|}(S\backslash|S|) = 1$. This is the extra $(1-\alpha)(m_{|S|} - m_{|S|-1})$ player $|S|$ receives when joining $S\backslash|S|$. It is the only time a player outside $S\backslash i$ shares in the $(1-\alpha)m_i$.

Our next two axioms compare bargaining strength across players. Consider two players in $S\backslash i$ with $j_1 < j_2$, competing for a share of $m_i$. Since $j_2$ has a weakly larger marginal contribution, it should get at least as much as $j_1$ gets on each interval $m_k - m_{k-1}$. We formalize this as Axiom 2.

**Axiom 2:** *Stronger players get weakly larger shares in the same competition: For $j_1, j_2 \in S\backslash i$ if $m_{j_1} \le m_{j_2}$, then $w_{j_2k}(S\backslash i) \ge w_{j_1k}(S\backslash i)$.*

It follows directly from Axiom 2 that players with equal strength get equal shares: For $j_1, j_2 \in S\backslash i$ if $m_{j_1} = m_{j_2}$, then $w_{j_1k}(S\backslash i) = w_{j_2k}(S\backslash i)$.

In addition, a player's share should be (weakly) smaller when it is competes against stronger players. If $j_1$ and $j_2$ are competing and $j_2$'s marginal contribution increases, this should lead to (weakly) lower weights for $j_1$. Note that here we do not require $j_1 < j_2$.

**Axiom 3:** For $j_1, j_2 \in S\backslash i$, *$w_{j_1k}(S\backslash i)$ is a weakly decreasing function of $m_{j_2}$.*

**Theorem A.1: Under Axioms 1, 2, and 3, there are unique payoff-relevant weights that lead to the Shapley value:**

$$w_{jk}(S\backslash i) = \begin{cases} \frac{1}{|S|-k} & \text{for } j \neq i,\ k < |S|, \text{ and } j \ge Min(i,k); \\ 1 & \text{for } i = j = k = |S|; \\ 0 & \text{otherwise}. \end{cases}$$

By "payoff relevant," we mean that if two (or more) of the marginal products are equal, e.g., $m_k = m_{k-1}$, so that $m_k - m_{k-1} = 0$, the weights on this interval do not enter into the players' payoffs.

**Proof:** These are the weights in our $\alpha$-procedure. It follows from Theorem 1 that they lead to the Shapley value and it is clear that these weights satisfy Axioms 1, 2, and 3. We need only demonstrate that no other weights satisfying Axioms 1, 2, and 3 lead to the Shapley value.

Consider the average negotiation gain to player $j \in S$ *as* $S$ goes from size $|S| - 1$ to $|S|$. Either $j$ is on the outside and joins $S\backslash j$, or $j$ is on the inside and shares in the gains as the set grows from $S\backslash i$ to $S$.

By Axiom 1, the expected gain to player $j$ is:

$$\frac{1}{|S|}[\alpha m_j + \sum_{i \in S}(1-\alpha)\sum_{k=1}^{Min(i,j)} w_{jk}(S\backslash i)(m_k - m_{k-1})]. \tag{A4}$$

Note that $w_{ik}(S\backslash i) = 0$ (since $i$ is on the outside) except when $i = k = |S|$, which is why the $i$ term is included in the summation.

The Shapley value payoff for $j$ is a specific function that depends only on player $j$'s marginal contributions. Therefore, a necessary condition for a generalized weighting system to result in the Shapley value is

$$\alpha m_j + \sum_{i \in S}(1-\alpha)\sum_{k=1}^{Min(i,j)} w_{jk}(S\backslash i)(m_k - m_{k-1}) = m_j. \tag{A5}$$

Since the weights are independent of $\alpha$, we can assume $\alpha < 1$. Thus Equation (A5) implies

$$\sum_{i \in S}\sum_{k=1}^{Min(i,j)} w_{jk}(S\backslash i)(m_k - m_{k-1}) = m_j. \tag{A6}$$

It remains to show there is a unique set of payoff-relevant weights that satisfy Axioms 1–3 and Equation (A6). Consider the payoff to player 1. By Axiom 1, the only positive weights $w_{1k}(S\backslash i)$ are when $k = 1$ and $i \in S\backslash 1$. Thus Equation (A6) implies

$$\sum_{i \in S\backslash 1} w_{11}(S\backslash i) m_1 = m_1. \tag{A7}$$

If $m_1 = 0$, then the weights $w_{11}(S\backslash i)$ are not payoff-relevant. When $m_1 > 0$, Equation (A7) implies

$$\sum_{i \in S\backslash 1} w_{11}(S\backslash i) = 1. \tag{A8}$$

Consider a new negotiation with $\tilde{m}_k = m_1$ for all $k$, where we still require that the weights lead to the Shapley value. Thus the new weights must sum to $1$:

$$\sum_{i \in S\backslash 1} \tilde{w}_{11}(S\backslash i) = 1. \tag{A9}$$

Because the marginal contributions are all equal, Axiom 2 implies that the weights must be equal across $j$ for $i \in S\backslash 1$: $\tilde{w}_{j1}(S\backslash i) = \tilde{w}_{11}(S\backslash i)$ for $j \in S\backslash i$. Since the sum of weights across $j$ is always $1$,

$$\tilde{w}_{11}(S\backslash i) = \frac{1}{|S| - 1} \text{ for all } i \in S\backslash 1 \tag{A10}$$

Now increase $\tilde{m}_{|S|}$ up to $m_{|S|}$. By Axiom 3, that must weakly lower all $\tilde{w}_{11}(S\backslash i)$. But it cannot strictly lower any of them since they must sum to $1$ by Equation (A9).

Next increase $\tilde{m}_{|S|-1}$ up to $m_{|S|-1}$. Again by Axiom 3 that must weakly lower all $\tilde{w}_{11}(S\backslash i)$. But it cannot strictly lower any of them since they must sum to $1$ by Equation (A9). We continue raising each of the $\tilde{m}_k$ up to $m_k$. At the end of this process, all $\tilde{m}_k = m_k$. Yet none of the $\tilde{w}_{11}(S\backslash i)$ have changed. Thus, for generalized weights that lead to the Shapley value and satisfy our axioms, a necessary condition is:

$$w_{11}(S\backslash i) = \tilde{w}_{11}(S\backslash i) = \frac{1}{|S| - 1}. \tag{A11}$$

At this point, we have only solved for the bargaining weight of the weakest player in the first interval. But given the size of this weight, the other players cannot have larger weights in the first interval.

All the other players are stronger, and thus, by Axiom 2, it follows that they must have weights at least as large as player $1$. For $i \neq 1$:

$$w_{j1}(S\backslash i) \geq w_{11}(S\backslash i) = \frac{1}{|S|-1}. \tag{A12}$$

However, $\sum_{j \in S\backslash i} w_{j1}(S\backslash i) = 1$. This equality can hold only if each value of $w_{j1}(S\backslash i)$ attains its minimum. Thus for $i \in S\backslash 1$, $w_{j1}(S\backslash i) = \frac{1}{|S|-1}$ for all $j \in S\backslash i$.

We need a modified argument to show that when $i = 1$, $w_{j1}(S\backslash 1) = \frac{1}{|S|-1}$ for $j \in S\backslash 1$. We do so by first showing $w_{21}(S\backslash 1) = \frac{1}{|S|-1}$.

Consider the expected payoff to player 2. By Axiom 1, the only positive weights on $w_{2k}(S\backslash i)$ are when $k = 1, 2$. Thus the necessary condition in Equation (A6) becomes

$$\sum_{i \in S\backslash 2} w_{21}(S\backslash i)m_1 + \sum_{i \in S\backslash 2,\ i \geq 3} w_{22}(S\backslash i)(m_2 - m_1) = m_2. \tag{A13}$$

Let $\tilde{w}_{21}(S\backslash 1)$ be the solution when $\tilde{m}_2 = m_1$ and all other $\tilde{m}_i = m_i$. In that case, Equation (13) reduces to

$$\sum_{i \in S\backslash 2} \tilde{w}_{21}(S\backslash i) = 1. \tag{A14}$$

Since $\tilde{w}_{21}(S\backslash i) = \frac{1}{|S|-1}$ for all $i \neq 1$, it follows from Equation (A14) that $\tilde{w}_{21}(S\backslash 1) = \frac{1}{|S|-1}$ as well. As before, $\sum_{j \in S\backslash 1} \tilde{w}_{j1}(S\backslash 1) = 1$. Since the $\tilde{w}_{j1}(S\backslash 1)$ are weakly increasing in $j$, this can only hold if $\tilde{w}_{j1}(S\backslash 1) = \tilde{w}_{21}(S\backslash 1) = \frac{1}{|S|-1}$ for $j \in S\backslash 1$.

Next, increase $\tilde{m}_2$ up to $m_2$. By Axiom 3, this weakly reduces all $\tilde{w}_{j1}(S\backslash 1)$ for $j \geq 3$ and potentially raises $\tilde{w}_{21}(S\backslash 1)$. But none of the $\tilde{w}_{j1}(S\backslash 1)$ for $j \geq 3$ can fall, and nor can $\tilde{w}_{21}(S\backslash 1)$ rise while maintaining $\sum_{j \in S\backslash 1} \tilde{w}_{j1}(S\backslash 1) = 1$ with $\tilde{w}_{j1}(S\backslash 1)$ being weakly increasing in $j$. Thus $\tilde{w}_{j1}(S\backslash 1) = \dfrac{1}{|S|-1}$ for $j \in S\backslash 1$. Since $\tilde{m}_i = m_i$ for all $i$, it follows that $w_{j1}(S\backslash 1) = \tilde{w}_{j1}(S\backslash 1) = \dfrac{1}{|S|-1}$. At this point, we have shown there is a unique solution for $w_{j1}(S\backslash i)$ among weights that satisfy Axioms 1–3 and that lead to the Shapley value.

To solve for $w_{j2}(S\backslash i)$, we follow a similar path. We assume $m_2 > m_1$, since otherwise the weights are not payoff-relevant.[14] We know from the above that $w_{21}(S\backslash i) = \dfrac{1}{|S|-1}$, and so the first sum in Equation (A14) equals $m_1$. Subtracting $m_1$ from both sides and dividing by $m_2 - m_1$ yields

$$\sum_{i \in S\backslash 2,\, i \geq 3} w_{22}(S\backslash i) = 1. \tag{A15}$$

The argument proceeds as above. Consider a new negotiation with $\tilde{m}_1 = m_1 < m_2$ and $\tilde{m}_k = m_2$ for all $k \geq 2$. Because the marginal contributions of the relevant players are all equal, Axiom 2 implies that for $i \in S\backslash 2$, $i \geq 3$, the weights must be equal across $j$: $\tilde{w}_{j2}(S\backslash i) = \tilde{w}_{22}(S\backslash i)$ for $j \in S\backslash i$.[15] Since the sum of weights across $j$ is always $1$,

$$\tilde{w}_{22}(S\backslash i) = \frac{1}{|S|-2} \text{ for all } 3 \leq i \leq |S|. \tag{A16}$$

As before, increase $\tilde{m}_{|S|}$ up to $m_{|S|}$. That must weakly lower $\tilde{w}_{22}(S\backslash i)$ for all $i$. But it cannot strictly lower any of them since the $\tilde{w}_{22}(S\backslash i)$ must sum to $1$ by Equation (A15).

Next increase $\tilde{m}_{|S|-1}$ up to $m_{|S|-1}$. That, too, must weakly lower all $\tilde{w}_{22}(S\backslash i)$. But it cannot strictly lower any of them since they must sum to $1$ by Equation (A15). At the end of this process, all of the $\tilde{m}_k = m_k$. Yet none of the $\tilde{w}_{22}(S\backslash i)$ have changed. Thus

$$w_{22}(S\backslash i) = \tilde{w}_{22}(S\backslash i) = \frac{1}{|S|-2}. \tag{A17}$$

As before, we have only solved for one case—here, player $2$'s weight in the second interval. By Axiom 2, it follows that for $j \geq 3$,

[14] If $m_2 = m_1$, we would consider the next non-zero interval, $m_k > m_{k-1}$, in our argument below.

[15] Recall that $\tilde{w}_{22}(S\backslash 1)$ is not meaningful since there is no interval $2$ when player $1$ is on the outside. Similarly, $\tilde{w}_{22}(S\backslash 2) = 0$ since player $2$ is on the outside.

$$w_{j2}(S\backslash i) \geq w_{22}(S\backslash i) = \frac{1}{|S|-2}. \tag{A18}$$

However, $\sum_{j\in S\backslash i,\, j\geq 2} w_{j2}(S\backslash i) = 1$. This equality can only hold if each value of $w_{j2}(S\backslash i)$ attains its minimum. Thus for $i \geq 3$, $w_{j2}(S\backslash i) = \frac{1}{|S|-2}$ for all $2 \leq j \leq |S|$ with $j \neq i$.

For $i = 1$, $w_{j2}(S\backslash 1)$ is not payoff-relevant since there is no second interval. For $i = 2$, $w_{j2}(S\backslash 2) = \frac{1}{|S|-2}$ by a parallel argument to the $i = 1$ case for $w_{j1}(S\backslash 1)$.

For $x < |S|$, parallel arguments show $w_{xx}(S\backslash i) = \frac{1}{|S|-x}$ for $i > x$, and then that $w_{jx}(S\backslash i) = \frac{1}{|S|-x}$ for $j > x$ and $i > x$. The last step that $w_{jx}(S\backslash x) = \frac{1}{|S|-x}$ for $j > x$ and $i = x$ also follows the argument as above.

The remaining weight to be specified is $w_{|S||S|}(S\backslash |S|)$. The interval $(m_{|S|} - m_{|S|-1})$ only exists when $|S|$ is on the outside. By Axiom 1, no inside players have a claim on this interval and thus $w_{|S||S|}(S\backslash |S|) = 1$ as required. ■