Reevaluating the Shapley Value: A New Justification and Extension

Adam Brandenburger *   Barry Nalebuff †

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Abstract

Inspired by the bargaining procedure of Shapley (1953), we introduce a novel procedure in which the marginal contribution of the player joining a coalition is split in any proportion between that player and the members of the coalition being joined. Surprisingly, this more general procedure also leads to the Shapley value. It is the unique generalized procedure that does so under a priority and a monotonicity axiom. Departing from the random-ordering assumption underneath the Shapley value (and our generalized procedure) leads to a new extension of the Shapley value where weights on players depend endogenously on marginal contributions.

1 Introduction

The Shapley value (Shapley, 1953) results from an axiomatic approach to the allocation of value in a cooperative game. It has an elegant procedural implementation: let the grand coalition of all players be constructed by players joining one at a time in a random order. In each ordering, give all the marginal value created by the new player joining the coalition to that player. The expected value is the Shapley value.

In this procedure, which we will call the Shapley procedure, the "negotiation" between the new player and the existing players in a coalition involves an extreme division of value. The player who joins receives the full amount of the benefit created. This feature has been commented on by Brock (1992):

> A criticism of this [Shapley's] scheme is that a player is awarded his entire (utility) contribution to a coalition. This violates an intuitive symmetry whereby we would expect the player to receive only a portion of his contribution to the coalition.

While each of the possible orderings in the Shapley procedure leads to an division of value which may appear extreme and, at a more prescriptive level, unfair, the average across all the orderings is balanced and reasonable. Still, at an intuitive level, we think a procedure would be more attractive both descriptively and prescriptively were the gains more evenly balanced among the parties in each possible ordering.

In our procedure, the player joining the existing coalition shares the gains with the members of the coalition in the proportion $\alpha : 1 - \alpha$. When $\alpha = 1$, we recover the original Shapley procedure. In the focal case of $\alpha = 1/2$, the gains are split evenly. For all $0 \leq \alpha < 1$, the existing members benefit when an additional player joins. We think this more general procedure is both more natural and more likely to be "bought into" by the players.

We are not the first to consider procedures in which the joining player shares the gains with the other players. In Nowak and Radzik (1994). Ju, Borm and Rurs (2007), and Malawski (2013), the part of the contribution of the joining player that goes to the members being joined is divided evenly among them. Our innovation is to consider unequal allocations—in order to reflect the different bargaining powers among the players. For example, in our procedure, a player that adds no value (a dummy player) does not share in the allocation.

We allow members of a coalition to have different claims on the $1 - \alpha$ share to be distributed. This leads us to allocate the distribution in stages. We begin with equal division, but only up to the smallest claim among the members sharing the gains. After that point, the member with the smallest claim no longer shares in the division. We divide the remaining share equally among the remaining members, but only up to the second-smallest claim. This process continues until the amount to be distributed is exhausted or all claims are exhausted. If no member of the coalition has a claim to the full $1 - \alpha$ share, the residual reverts to the joining player. This procedure allows for what we think is a natural interplay between the

*Stern School of Business, Tandon School of Engineering, NYU Shanghai, New York University, New York, NY 10012, U.S.A., adam.brandenburger@stern.nyu.edu

†Yale School of Management, Yale University, New Haven, CT 06511, U.S.A., barry.nalebuff@yale.edu

forces of bargaining power and equal division. The equal division at each stage resembles the "principle of the divided cloth" (O'Neill, 1982; Aumann and Maschler, 1985). We return to this connection in the next section.

We are interested in the case where the amount to be distributed is the marginal contribution of the joining player and the claims equal the marginal contributions of the players being joined, where the marginal contributions are to the full set consisting of the existing members and the joining player. We find, perhaps surprisingly, that this procedure yields the Shapley value for all α. Indeed, it is the unique procedure that does so (under conditions provided in Section 5). We see our findings as providing additional understanding of and support for the Shapley value. It shows that the Shapley value arises under more general circumstances, and, arguably, more intuitively appealing ones. It also highlights that what is essential to obtaining the Shapley value is the random ordering of how players join existing coalitions, not the allocation of all the marginal contribution to the joining player. In Section 7, we allow the ordering to depend on marginal contributions; this leads to a new family of generalized Shapley values, related to but different from the well-known weighted Shapley value (Shapley, 1953).

Dividing up the $1 - \alpha$ share among the members of an existing coalition suggests that each member's payoff will depend on the marginal contributions of all the players in the coalition. But the Shapley value payoff to a player depends only on that player's marginal contributions. (This fact characterizes the Shapley value; see Young (1985).) We still get the Shapley value because these extra dependencies cancel out. Each player obtains less than their marginal contribution when joining a coalition. But this shortfall is exactly offset by the extra amounts each player collects when other players join all the coalitions to which that player belongs.

A procedure is a way of understanding a cooperative solution concept, but it is not a non-cooperative implementation of the concept. We do not see this as a shortcoming of our analysis. A procedure is playable in an intuitive sense, where the players agree to various divisions of value and do so without an exact offer-counteroffer protocol that a non-cooperative analysis would require. Nevertheless, it is possible to provide a fully non-cooperative game that mimics our procedure and builds on the non-cooperative implementation of the Shapley value in Hart and Mas-Colell (1996).

The Shapley procedure and our procedure build up coalitions by adding an outside player to each existing coalition (in a random order). We also consider a reverse procedure in which we start with the full set of players and then potentially exclude players one at a time (in a random order). For each random construction of a coalition, there is an equivalent deconstruction: instead of players joining in order ABC, say, we can think of first C, then B, then A contemplating being excluded. The reverse procedure can be understood as a scenario in which all players are present at the start and consider what might happen if they fail to reach an agreement. The players agree to accept the expected value of the exclusion procedure. That expectation is again the Shapley value. The reverse procedure is closest to the non-cooperative implementation of the Shapley value, to be reviewed later.

We begin in Section 2 with axioms that define our general procedure for a given set of claims and a division rule α. Section 3 connects the claims to the marginal contributions of the players and offers a numerical example that illustrates how our procedure leads to the Shapley value. Section 4 proves that our procedure yields the Shapley value. Section 5 establishes that our procedure is unique within a family of procedures satisfying two axioms. Section 6 modifies our procedure to run in reverse—a procedure that points to a new generalized Shapley value. In Section 7, we compare our generalized Shapley value with the weighted Shapley value (Shapley, 1953; Kalai and Samet, 1987). Section 8 contains further discussion of related literature, a comment on proportional solutions, an outline of a non-cooperative implementation of our procedure, and a brief conclusion.

2 Three Axioms

Fix a set of players S. Consider the allocation when player $i \in S$ joins $S \backslash i$. In the Shapley procedure, the marginal contribution created by the joining player all goes to that player. Here, the marginal contribution is divided up in proportion $\alpha : 1 - \alpha$. Share α goes to the joining member and share $1 - \alpha$ goes with priority to members of the existing coalition. When $\alpha = 1$, the two procedures coincide.

When $\alpha < 1$, we have to specify how the $1 - \alpha$ share is allocated among the members of the existing coalition. We use as a springboard the principle of the divided cloth (O'Neill, 1982; Aumann and Maschler, 1985). Players whose claims exceed the amount being distributed share equally. Players whose claims are below this amount share equally up to the point of their respective claims. This approach can be viewed as a multi-player version of the principle of the divided cloth—one that takes into account the claims of the players being joined and that of the joining player. The guiding principle is equal division constrained by claims.

To bring out the essential features of our approach, we begin with an abstract framework. Let $d \geq 0$

denote the amount to be divided and let each player $j \in S$ have a claim c_j. We assume as a convention that the c_j are ordered so that $c_1 \leq c_2 \leq \ldots \leq c_{|S|}$, and we define $c_0 = 0$. If there are tied claims, they are ordered arbitrarily. Once this ordering is fixed, the claims are fixed for a set S and do not vary with the relevant subset of players. Thus, player 1's claim is always c_1, whether player 1 is part of the coalition $S\backslash 2$, $S\backslash 3$, or of any other coalition $S\backslash i$ with $i \in S\backslash 1$. When player 1 is the joining player, that player's claim is still c_1, but, as we will see, that claim is given lower priority than the claims of the players in $S\backslash 1$.

Our division coincides with the principle of the divided cloth when there are two players in $S\backslash i$ and at least one of these players has a claim on the full amount.

Example 1. *Let $S = \{1, 2, 3\}$. Consider the division between player 1 and player 2 when player 3 joins $S\backslash 3$ and distributes d. We assume $d = 2$, and player 1 (resp. player 2) has a claim of 1 (resp. 2). The first unit of d is claimed by both players and is split equally, while the second unit of d all goes to player 2. The resulting division of d is $(1/2, 3/2)$ between player 1 and player 2.*

In cases with more than two players in $S\backslash i$, we extend our logic as follows. Any amounts commonly claimed by a subset of players in $S\backslash i$ are shared equally by that subset. Any amount above a player's claim goes to the players who do have a claim. This is a different treatment of the n-player case from the consistency requirement in Aumann and Maschler (1985), and it is therefore unsurprising that it leads to a different solution concept. Aumann and Maschler obtain the nucleolus (Schmeidler, 1969), while we obtain a different but also classic solution concept, namely, the Shapley value.

To define our framework formally, fix a player $i \in S$. This is the "joining" player. Consider the division of the total amount d among the members of the set being joined, namely, $S\backslash i$, given the various $|S| - 1$ claims they have. Let $j^*(i)$ denote the largest number $j \in S\backslash i$ such that $c_j < d$. We partition d into intervals:

$$\{[c_0, c_1), [c_1, c_2), \ldots, [c_{j-1}, c_j), \ldots, [c_{j^*(i)}, d]\}. \tag{1}$$

Note that since $i \notin S\backslash i$, the claim c_i is not the lower limit of any interval in (1). Either $c_i \geq d$, in which case the partition ends before getting to an interval with lower limit c_i. Or, $c_i < d$, in which case the partition skips the interval beginning with c_i.

Example 2. *Suppose $i = 2$, $c_i < d$, and $j^*(i) = 3$. The partition is then $\{[c_0, c_1), [c_1, c_3), [c_3, d]\}$.*

We assume that $d \leq c_{|S|}$, so that at least one player (potentially, player i) has a claim on the full amount. As we will see, this assumption will always be satisfied in our procedure. We can think of the amount d being distributed in the form of a "flow." Initially, the flow is directed equally to all $|S| - 1$ members of $S\backslash i$. As soon as the amount distributed exceeds a player's claim, that player no longer shares in the flow going forward. The flow is now directed, again equally, to the remaining $|S| - 2$ players of $S\backslash i$. And so on. This process continues until there is nothing left to distribute or there is some unclaimed amount left over. Any unclaimed amount reverts to the joining player i. Figure 1 illustrates the process in the first case and Figure 2 depicts the second case. The formal statement is the following two axioms.

> **Interval Equality Axiom:** *Each interval $[c_{j-1}, c_j)$ is divided equally among all players in $S\backslash i$ whose index is greater than or equal to j. The final interval $[c_{j^*(i)}, d]$ is divided equally among all players in $S\backslash i$ whose index is greater than or equal to $j^*(i) + 1$, assuming that set is non-empty. Otherwise, the final interval is not allocated to the players in $S\backslash i$.*

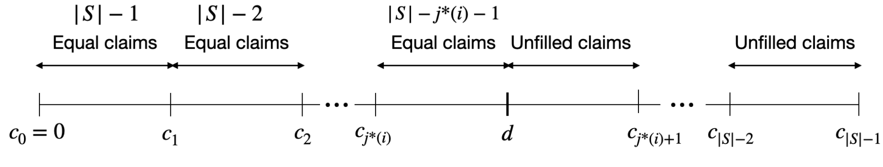

Figure 1: Interval Equality Axiom

When no player in $S\backslash i$ has a claim on the full amount d, the unclaimed portion reverts to the joining player i. This case arises only if player i is the unique member of S whose claim is equal to or greater than d, that is, $i = |S|$ and $j^*(i) = |S| - 1$.

> **Priority Axiom:** *If $d > c_{|S|-1}$, the interval $[c_{|S|-1}, d]$ is allocated to player i.*

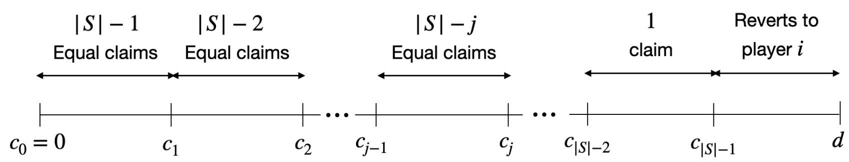

Figure 2: Priority Axiom

We call this the Priority Axiom because the claims c_j for players $j \in S\backslash i$ are prioritized over the claim c_i of player i. If any member of the coalition being joined has a claim on the full amount to be distributed, then everything goes to this coalition. Only when $i = |S|$ and $c_{|S|-1} < d \leq c_{|S|}$ is there a (non-empty) interval $[c_{|S|-1}, d]$. This amount is unclaimed by members of $S\backslash i$, and the joining player i's claim on this interval is then satisfied.

The Priority Axiom is an essential feature of any 1-player game. Player 1 always joins the null set. There are no players in the null set and thus no existing player has any claim on d. By assumption, some player has a full claim on d, so it must be Player 1 in this game. Thus, d is fully allocated to the joining player. Absent the Priority Axiom, we would have an inefficient result in any 1-player game. Similarly, when $S\backslash i$ is non-empty, if there is some unclaimed portion of d, the Priority Axiom ensures that the allocation is always efficient.

Example 3. *Let* $S = \{1, 2, 3, 4\}$. *Suppose* $d = 6$, *the claims in* S *are* $c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 6$, *and player 4 is the joining player. Here the partition according to expression 1 is* $\{[0, 1), [1, 2), [2, 5), [5, 6]\}$, *and so* $j^*(4) = 3$. *The allocations to the players in* $S\backslash 4$ *from first two intervals are* $(1/3, 1/3, 1/3)$ *and* $(0, 1/2, 1/2)$. *The third interval* $[2, 5)$ *goes entirely to player 3. Player 4 gets no allocation from these intervals, since the claims of the players being joined are prioritized. However, none of the three players being joined has a claim to the interval* $[5, 6]$, *which therefore reverts to player 4. The resulting allocation is* $(1/3, 1/3 + 1/2, 1/3 + 1/2 + 3, 1)$.

We introduce a final axiom concerning the division between the player joining a coalition and the coalition being joined. The parameter α describes the bargaining power of the player joining the group. When $\alpha = 1$, the joining player does not share anything with the players in $S\backslash i$. When $\alpha = 0$, the entire amount d is allocated to the players in $S\backslash i$—except when there is an unclaimed amount and the Priority Axiom comes into play. For $0 < \alpha < 1$, both the amount to be distributed and the claims by the coalition being joined are scaled by $1 - \alpha$.

Note that α is fixed for a given set S. In particular, the bargaining power α does not vary with the player i joining $S\backslash i$. However, we do allow α to vary across sets S. As an example, $\alpha = 1/|S|$ is permitted, in which case the bargaining power of the coalition $S\backslash i$ is proportional to its size.

Power Axiom: For a given bargaining parameter α, the amount αd goes to the joining player i, and the remaining amount $(1 - \alpha)d$ is distributed according to the Interval Equality and Priority Axioms, where claims are also scaled by $1 - \alpha$.

Example 3 contd. *Continuing the previous example, if* $\alpha = 1/2$, *then* $1/2 \times 6 = 3$ *will go directly to the joining player. The other* 3 *units follow the prior allocation, just multiplied by 1/2. The adjusted claims are* $(1/2, 1, 5/2, 3)$ *for the 3 units to be distributed. The adjusted partition is* $\{[0, 1/2), [1/2, 1), [1, 5/2), [5/2, 3]\}$, *and the resulting total allocation is* $(1/6, 1/6 + 1/4, 1/6 + 1/4 + 3/2, 3 + 1/2)$.

3 Specification of Claims

In the setting of the principle of the divided cloth, claims are exogenous. In this paper, the claims are endogenous to the cooperative game. We are interested in the claims made by the members of a coalition S, as different members $i \in S$ join $S\backslash i$. As in the prior section, claims are fixed for a fixed set S; they do not vary with who joins and who is being joined. We adopt core-like logic Gillies (1959) and take each player's claim to be equal to that player's marginal contribution to S. This assumption implies that only members of S share in the gains. The core argument is that a player who claims more than their marginal contribution will be excluded by the other members of S.

Let N be the player set. Associated with each coalition $S \subseteq N$ is a real number $v(S) \geq 0$, namely, the value created by the players in S. We assume that v is superadditive, that is, $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$. Let $m_i(S) = v(S) - v(S\backslash i)$ denote the marginal contribution of player i to S. Superadditivity implies $m_i(S) \geq 0$. We add a fictitious player 0 who is not a member of any coalition and set $m_0(S) = 0$ for all S in order to simplify later equations.

We adopt the following labeling convention: for each S, order the players according to their marginal contributions to S, so that $m_0(S) \leq m_1(S) \leq \cdots \leq m_{|S|}(S)$. If two players have equal marginal contributions, order them arbitrarily. This ordering will generally be different across different sets S. When the meaning is clear, we write m_i for $m_i(S)$ in order to reduce notation.

Amount Divided: Set $d = m_i(S)$ when player i joins $S\backslash i$.

Claims: Set $c_j = m_j$ for all $j \in S$.

The c_j inherit the weakly increasing order of the m_j and so are ordered as in the previous section. An immediate implication of our specification of claims is that a dummy player will have a claim of 0, and so, by our Interval Equality Axiom, will always receive 0. A implication of the two specifications together is that $d \leq m_{|S|} = c_{|S|}$, as we assumed in the previous section. Player $|S|$ will always have a claim to the full amount of d. It follows that there will be no reversion to the joining player whenever player $|S| \in S\backslash i$. Reversion takes place only when $|S|$ is the joining player.

The following three-player example illustrates how our three axioms work together and also provides intuition for why our procedure gives each player their Shapley value.

Example 4. *Consider a three-player coalition S where $m_1 < m_2 < m_3$. Suppose first that $\alpha = 0$ so that the Power Axiom grants 0 to the joining player. When player 1 is the joining player, it receives 0 because the amount $d = m_1$ is fully claimed by players 2 and 3. When being joined, player 1's claim is limited to m_1. It splits this amount equally with either player 2 (when player 3 joins) or player 3 (when player 2 joins). Thus, the average amount player 1 receives is $1/3 \times [0 + m_1/2 + m_1/2] = m_1/3$. When $\alpha = 1$, player 1 gets its entire marginal contribution m_1 when it is the joining player and 0 otherwise: $1/3 \times [m_1 + 0 + 0] = m_1/3$. The average payout to player 1 is the same in both cases.*

When $\alpha = 0$ and player 2 is the joining player, it also receives 0 directly and 0 under the Priority Axiom. When being joined, player 2's claim is limited to m_2. When player 1 is the joiner, the total to be distributed is m_1, which is claimed by both players 2 and 3, and thus split equally. When player 3 is the joiner, the total to be distributed is m_3. Player 2 splits m_1 equally with player 1 and also receives the full amount $m_2 - m_1$, at which point player 2's claim is exhausted. The remaining $m_3 - m_2$ reverts to player 3. The average amount player 2 receives is $1/3 \times [0 + m_1/2 + m_1/2 + (m_2 - m_1)] = m_2/3$, the same as when $\alpha = 1$.

Finally, when $\alpha = 0$ and player 3 is the joining player, it receives $m_3 - m_2$ under the Priority Axiom. When player 1 is the joiner, player 3 splits m_1 equally with player 2. When player 2 is the joiner, player 3 splits m_1 equally with player 1 and also receives the full amount $m_2 - m_1$. The average amount player 3 receives is $1/3 \times [(m_3 - m_2) + m_1/2 + m_1/2 + (m_2 - m_1)] = m_3/3$, once again the same as when $\alpha = 1$.

In this example, all players receive on average the same amount when $\alpha = 0$ as when $\alpha = 1$. But the case $\alpha = 1$ is exactly the Shapley procedure (where the joining player receives the full marginal contribution). The Power Axiom implies that, for intermediate values of α, the payoff received by each player is the $\alpha : 1-\alpha$ weighted average of what that player receives at $\alpha = 1$ and $\alpha = 0$. Since these two payoffs are the same, each player receives the same payoff for all values of α. Moreover, this constant payoff is the Shapley value. In the next section, we show that this result holds in full generality.

4 The α-Procedure

We now analyze a general transferable-utility (TU) cooperative game under our α-procedure and show that the expected payoffs to the players are independent of α. This will establish that our procedure yields the Shapley value for all α.[1]

Fix a player set N. The α-procedure works as in Shapley (1953) by considering all random orderings of the set N, which induce all random orderings of any coalition $S \subseteq N$.[2] We focus on the $|S|$ possible orderings for the last step in how coalition S is formed. For each pair of players $(i, j) \in S$, let $\pi_j(S, i; \alpha)$ be player j's payoff when i joins $S\backslash i$. We allow $i = j$. This payoff is determined by the bargaining parameter α and our three axioms. Because orderings are random, the average value of $\pi_j(S, i; \alpha)$ over the $|S| - 1$ equally probable cases where player j is being joined and the one case in which j is the joiner is given by

$$\pi_j(S; \alpha) = \frac{1}{|S|} \sum_{i \in S} \pi_j(S, i; \alpha). \tag{2}$$

The Shapley procedure coincides with the α-procedure when $\alpha = 1$. Each player j receives $m_j(S)$ when joining $S\backslash j$ and 0 in the other $|S| - 1$ cases where a player $i \in S\backslash j$ joins j. Thus $\pi_j(S; 1) = m_j(S)/|S|$. In the general case, each player $j \in S$ receives $\alpha m_j(S)$ when joining $S\backslash j$ and, when player i joins $S\backslash i$, a share of the amount $(1 - \alpha)m_i(S)$.

To specify the division of the amount $(1 - \alpha)m_i$ among the $|S| - 1$ players in $S\backslash i$ (and to player i as well, when there is an unclaimed amount), we partition the amount m_i just as in the partition (1) in Section 2:

$$\{[m_0, m_1), [m_1, m_2), \ldots, [m_{j-1}, m_j), \ldots, [m_{j^*(i)}, d]\}, \tag{3}$$

[1]Brandenburger and Nalebuff (2024b) extends the α-procedure to NTU games.

[2]In Section 7, we consider a generalization in which arrival order (or coalition formation) depends on the value created.

where $d = m_i$. The starting indices on the intervals comprising the partition are consecutive in number, from 0 to $j^*(i)$. In particular, the issue of skipping the interval whose lower limit is m_i does not arise because $c_i = m_i = d$ and thus $c_i \geq d$. Also, note that we can equally well write the final interval as $[m_{j^*(i)}, m_{j^*(i)+1}]$ since, again using $c_i = m_i = d$, we must have $m_{j^*(i)+1} = d$.[3]

The Interval Equality Axiom says that when player i joins $S\backslash i$, player $j \in S\backslash i$ receives a $1/(|S| - k)$ share of the interval with ending index k. This allocation to j holds for intervals up to the smaller of j (since its maximum claim is m_j) and $j^*(i) + 1$ (the number of intervals into which the total amount m_i is partitioned). Thus for $j \neq i$:

$$\pi_j(S, i; \alpha) = (1 - \alpha) \sum_{k=1}^{\min\{j, j^*(i)+1\}} \frac{m_k - m_{k-1}}{|S| - k}. \tag{4}$$

When player j joins $S\backslash j$, it receives αm_j. In addition, when $j = |S|$, the Priority Axiom dictates that j also receives $(1 - \alpha)(m_{|S|} - m_{|S|-1})$. Putting everything together, the α-procedure specifies that when player i joins $S\backslash i$, player j receives a payoff $\pi_j(S, i; \alpha)$ given by:

$$\pi_j(S, i; \alpha) = \begin{cases} (1 - \alpha) \displaystyle\sum_{k=1}^{\min\{j, j^*(i)+1\}} \frac{m_k - m_{k-1}}{|S| - k} & \text{if } j \neq i; \\ \alpha m_j & \text{if } j = i < |S|; \\ \alpha m_{|S|} + (1 - \alpha)(m_{|S|} - m_{|S|-1}) & \text{if } j = i = |S|. \end{cases} \tag{5}$$

Theorem 1. *In the α-procedure, $\pi_j(S; \alpha) = \pi_j(S; 1)$ for all α in $[0, 1]$.*

Proof. Fix a coalition S. We first show that $\pi_j(S, i; \alpha) = \pi_i(S, j; \alpha)$ for $i, j \in S$. This is immediate if $j = i$, so suppose $j \neq i$. We examine Equation 4 in the two possible cases: (i) $m_{i-1} < m_i$, and (ii) $m_{i-1} = m_i$. In case (i), we have $j^*(i) = i - 1$, so that $\min\{j, j^*(i) + 1\} = \min\{j, i\}$, which is symmetric in i and j. In case (ii), we have $j^*(i) < i - 1$. In this case, too, we replace $j^*(i)$ with $i - 1$, so that the upper limit of the sum in Equation 4 is $\min\{j, i\}$ as in case (i). In making this substitution, we potentially add extra terms to the sum. The numerators in these extra terms start at $m_{j^*(i)+2} - m_{j^*(i)+1}$, and, depending on the value of j, they potentially go up to $m_i - m_{i-1}$. By the definition of $j^*(i)$, we have $m_{j^*(i)+1} = \ldots = m_i$. It follows that the numerators $m_k - m_{k-1}$ in the range $k = j^*(i) + 2, \ldots, i$ are all equal to 0, and thus the potential addition of terms to the sum in Equation 4 does not change its value. In both cases (i) and (ii), we can replace Equation 4 with:

$$\pi_j(S, i; \alpha) = (1 - \alpha) \sum_{k=1}^{\min\{j, i\}} \frac{m_k - m_{k-1}}{|S| - k} = \pi_i(S, j; \alpha), \tag{6}$$

which is symmetric in i and j.

Now consider the expected payoff $\pi_j(S; \alpha)$ to player $j \in S$. We have:

$$\pi_j(S; \alpha) = \frac{1}{|S|} \sum_{i \in S} \pi_j(S, i; \alpha) = \frac{1}{|S|} \sum_{i \in S} \pi_i(S, j; \alpha) = \frac{m_j(S)}{|S|}. \tag{7}$$

Here, the second equality comes from interchanging i and j. The third equality holds because, for all α, player j's marginal contribution is fully allocated across the full set of players $i \in S$. We conclude that $\pi_j(S; \alpha)$ is independent of α, from which $\pi_j(S; \alpha) = \pi_j(S; 1)$ for all α, as required. □

The idea of the proof of Theorem 1 is that what a player gives up when joining a coalition is exactly offset by what that player gets when it is inside the coalition and each of the other players joins. Consider, for example, player 1. It gives up $(1 - \alpha)m_1$ when it joins the coalition. But, by the Interval Equality Axiom, it gets back $(1 - \alpha)m_1/(|S| - 1)$ a total of $|S| - 1$ times—once for each time any of the other $|S| - 1$ players joins it in the coalition. The loss and gains exactly offset each other. The proof establishes that a similar argument holds for each of the other players. The Priority Axiom takes care of the case when the joining player has the largest marginal contribution, so that the claims inside the coalition do not exhaust the amount to be distributed.

Corollary 1.1. *In the α-procedure, the expected payoff to each player is the Shapley value for all α in $[0, 1]$.*

Proof. When $\alpha = 1$, each player $j \in S$ receives m_j when joining $S\backslash j$ and receives 0 when $i \in S$ joins $S\backslash i$. This case coincides with the Shapley procedure. By Theorem 1, player j receives the same expected payoff for all α. □

[3] If $m_{i-1} < m_i$, then $j^*(i) + 1 = i$ and the final interval is $[m_{i-1}, m_i]$. Otherwise, $j^*(i) + 1 < i$, and the upper limit of the final interval has a value equal to m_i, even though the ending index is less than i.

5 Uniqueness

Our α-procedure is the unique procedure within a general class of procedures that yield the Shapley value and satisfy our Priority Axiom and a Monotonicity Axiom.[4] The Monotonicity Axiom, formalized below, requires that players with larger marginal contributions receive at least as much as those with smaller marginal contributions.[5]

We now allow for a more general division of the $1-\alpha$ share of value that goes to the members of the coalition being joined. As before, when player i joins $S\backslash i$, its marginal contribution is partitioned into $j^*(i)+1$ intervals. When the marginal contributions are distinct, $j^*(i)+1=i$. Since a general weighting system has to allow for that possibility, we partition m_i into i intervals (and recognize that intervals past $j^*(i)+1$ will be of length 0).

Under the Interval Equality Axiom, each interval is divided equally among the players with a claim to that interval. Here we allow for unequal divisions. For each $k \leq i$, we specify weights $w_j(S,i;k)$ that determine player j's share of the kth interval, $[m_{k-1}, m_k)$, when player i joins $S\backslash i$. If two or more players are tied, we order them arbitrarily, as before. To support the interpretation in terms of weights, we require for each $k \leq i$:

$$\sum_{j \in S} w_j(S,i;k) = 1. \tag{8}$$

The Priority Axiom still applies. Thus, the players being joined receive all of $(1-\alpha)m_i$, except in the case $i=|S|$. This implies that $w_j(S,j;k)=0$ for all j,k except when $j=k=|S|$, in which case $w_j(S,j;k)=1$.

The Interval Equality Axiom specifies: $w_j(S,i;k)=\frac{1}{|S|-k}$ for $j \neq i$ and $k \leq \min\{j,i\}$. For this set of $\{i,j,k\}$, the weights in each interval are constant across i and j. Here, the weights in each interval can vary with i and j, and a player can share in intervals beyond its claim. For example, weights can be proportional to the player's rank in the ordering. Formally, for $j \neq i$ and $k \leq i$:

$$w_j(S,i;k) = \frac{j}{|S|(|S|+1)/2 - i}. \tag{9}$$

In this case, player j's weights are positive even for intervals $k > j$.

Alternatively, weights could be proportional to the player's rank, but the division is limited to the players who have a claim on a given interval. (This ensures a dummy player receives 0.) Formally, for $j \neq i$ and $k \leq i$:

$$w_j(S,i;k) = \begin{cases} 0 & \text{if } j < k; \\ \frac{j}{(|S|+1-k)(|S|+k)/2-i} & \text{if } j \geq k. \end{cases} \tag{10}$$

Another possibility is that all the weight could go to the player in $S\backslash i$ with the highest rank. Formally, for $j \neq i$ and $k \leq i$:

$$w_j(S,i;k) = \begin{cases} 1 & \text{if } j = \max\{S\backslash i\}; \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

It can be checked that each of these examples satisfies Equation 8 for $k \leq i$, as required. It is also true that in each of these examples, the weights are weakly increasing with a player's rank. Our Monotonicity Axiom requires that this holds in general.

Monotonicity Axiom: The weights $w_j(S,i;k)$ are weakly increasing in $j \in S\backslash i$.

Intuitively, if player j's marginal contribution to S is larger than that of player j', thenplayer j is more powerful than player j' and should therefore receive more value in each interval $[m_{k-1}, m_k)$.

We denote the weights in our α-procedure by $w_j^*(S,i;k)$:

$$w_j^*(S,i;k) = \begin{cases} \dfrac{1}{|S|-k} & \text{if } j \neq i, k \leq \min\{i,j\}; \\ 1 & \text{if } i=j=k=|S|; \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

These weights satisfy the Interval Equality and Priority Axioms, and by Theorem 1 and Corollary 1.1, they yield the Shapley value as payoffs. Interval Equality is stronger than Monotonicity and therefore implies the Monotonicity axiom is satisfied. While other weighting systems satisfy the Priority and Monotonicity Axioms, only one such weighting scheme always leads to the Shapley value.

Theorem 2. *Under the Priority and Monotonicity Axioms, the weights $w_j^*(S,i;k)$ of the α-procedure uniquely yield the Shapley value.*

[4]We thank Sergiu Hart for encouraging us to consider this question.

[5]The monotonicity is across players. This is different from Young's (1985) monotonicity axiom, which operates across games.

The proof of Theorem 2 is based on a repeated application of Unanimity Games, one for each interval k. The details are straightforward, but lengthy, and can be found in the online appendix (Brandenburger and Nalebuff, 2024a).

Absent our Monotonicity Axiom, there are other weights that lead to the Shapley value, as the following example shows.

Example 5. *Let $S = \{1, 2, 3\}$ and fix the weights $w_j(S, i; 1)$ for the first interval to be:*

$$w_1(S, 2; 1) = 1,\ w_3(S, 2; 1) = 0, \tag{13}$$
$$w_1(S, 3; 1) = 0,\ w_2(S, 3; 1) = 1, \tag{14}$$
$$w_2(S, 1; 1) = 0,\ w_3(S, 1; 1) = 1, \tag{15}$$

with all other weights given by Equation 12. Focus on the first interval. When player 2 joins $\{1, 3\}$, player 1 receives $(1 - \alpha)m_1$ and player 3 receives 0. When player 3 joins $\{1, 2\}$, player 1 receives 0 and player 2 receives $(1-\alpha)m_1$. When player 1 joins $\{2, 3\}$, player 2 receives 0 and player 3 receives $(1-\alpha)m_1$. Summing across the three cases, each player receives a total of $(1-\alpha)m_1$, which is the same as under the α-procedure. All other payoffs are also the same as under the α-procedure, since the weights are the same. The result is that each player receives their Shapley value.

In our view, the counterintuitive aspect of this example is that when player 2 joins $\{1, 3\}$, the weaker player 1 receives the full amount $(1-\alpha)m_1$, while the stronger player 3 receives 0. Of course, this also shows that the weights in Example 5 do not satisfy our Monotonicity Axiom.

6 A Reverse Procedure

Our α-procedure follows the Shapley procedure in that players join one at a time as the player set is built up to obtain the grand coalition N. In this section, we start with the grand coalition and examine what might happen if the players fail to reach an agreement on dividing the total value $v(N)$. In the reverse procedure, the players bargain in the "shadow" of an elimination process that operates if they cannot agree on a division.

We believe the reverse procedure is of intrinsic interest, but we also see two additional reasons to study it. First, the reverse procedure suggests a new generalization of the Shapley value; see Section 7. Second, the reverse procedure naturally suggests a non-cooperative implementation of the Shapley value, analogous to the one in Hart and Mas-Colell (1996); see Section 8.3.

In our reverse procedure, we assume that if agreement over the division of $v(N)$ is not reached, then a player $i \in N$ is chosen at random to be at risk of being eliminated from the grand coalition and thereby getting 0 in any subsequent division. This player i bargains with the other players over the payoff it will accept in order to remain in the grand coalition. The reverse procedure determines this payoff by assigning player i an α share of its marginal contribution to $v(N)$, while the other players split the $1 - \alpha$ share of this marginal contribution in exactly the same fashion as in our earlier α-procedure. Letting $\pi^j(N; \alpha)$ denote the resulting expected payoff to player j, we have:

$$\pi^j(N; \alpha) = \frac{1}{|N|} \sum_{i \in N} \left[\pi_j(N, i; \alpha) + \pi^{j(i)}(N \backslash i; \alpha) \right], \tag{16}$$

where $\pi_j(N, i; \alpha)$ specifies the $\alpha : 1 - \alpha$ split just as in Equation 5 (applied to $S = N$) and $\pi^{j(i)}(N \backslash i; \alpha)$ is the expected payoff that player j, ranked j in N and ranked $j(i)$ in $N \backslash i$, anticipates in the game without i.

To determine $\pi^j(N \backslash i; \alpha)$, we repeat the hypothetical elimination process. A player $h \in S \backslash i$ is chosen at random to be at risk of elimination. There is an $\alpha : 1 - \alpha$ division of player h's marginal contribution to $S \backslash i$, where the split is now between h and $S \backslash \{i, h\}$. This hypothetical elimination process next moves to the set of players $S \backslash \{i, h\}$, and it continues in analogous fashion until only one player remains.

For each order of potential elimination of players under the reverse procedure, the expected payoffs to all players are identical to the expected payoffs in our previous α-procedure—but where the order of joining is the reverse of the order of elimination. For example, suppose the order of potential elimination is player 1→player 2→player 3 (where the players are labelled according to their rank in N). Then the expected payoffs are the same as under the α-procedure when the order of joining is player 3→player 2→player 1. In both cases, player 1 bargains with the coalition $\{2, 3\}$ and player 2 bargains with player 3. With all $|N|!$ orderings equally likely, the expected payoffs will be the same whether the procedure is run forward or in reverse. The expected payoffs under our reverse procedure therefore coincide with those under the α-procedure.

7 Generalized Shapley Values

Shapley (1953) and Kalai and Samet (1987) extended the original Shapley value to incorporate weights that depend on the identity of a player. These weights can be understood as influencing the order in which players join existing coalitions. Our reverse procedure suggests a different way to generalize the Shapley value via weights. In the reverse procedure, we supposed that all players face an equal chance of elimination at each stage. More generally, some players might face a higher probability of elimination than others. For example, if a coalition breaks down, it might do so in the least expensive way—meaning that a player with the smallest marginal contribution would be the first to be at risk of elimination.

This idea that some coalitions are more likely to form than others builds on Aumann and Drèze (1974, Section 12.6) and Owen (1977). Some coalitions are imagined to form first because of the large value they create for their members. We differ from Aumann and Drèze (1974) in that they allow a coalition to break up in the event another player joins the bargaining. Owen (1977) assumes that when a coalition bargains with the joining player, the gain is split evenly, which is akin to setting $\alpha = 1/2$ in our setting. The division of value inside the coalition being joined sets the payoff differences across the players according to the Shapley value. Thus, Owen employs the Shapley value as an input to the division procedure.

Our generalized approach employs the same division method as in the α-procedure. The difference is only in the order of coalition formation. An example adapted from Littlechild and Owen (1973) makes the comparison clear.

Example 6. *Three airlines share the cost of a common runway. Suppose airline 1 requires a runway of length* 1*, while airlines 2 and 3 each require a runway of length* 2*. The cost per unit length is* 1*. If all airlines are equally likely to be at risk of elimination, then airline 1 earns a payoff of* α *when it is the first to be at risk,* $(1-\alpha)/2+\alpha$ *when it is the second at risk, and* $(1-\alpha)/2+(1-\alpha)$ *when it is the last to be at risk. On average, airline 1 earns* $2/3$*. This is its Shapley value—a fact we could deduce from Corollary 1.1 and the observation that, when ordering is random, the reverse procedure yields the same expected payoffs as the* α*-procedure.*

Now consider a weighted variant of this game in which if the coalition $\{1,2,3\}$ *breaks down, airline 1 is certain to be the first at risk of elimination. This case is of interest because less value is lost if airline 1 is eliminated as compared with what happens if airline 2 or airline 3 are eliminated. This line of argument leads to two equally likely elimination orderings: airline 1→airline 2 and airline 1→airline 3. The resulting expected payoffs to airlines 1, 2, and 3 are* α*,* $(1-\alpha)/2+1$*, and* $(1-\alpha)/2+1$*, respectively. Airline 1's expected payoff is the same as in the first component of the equal-likelihood case above. The remaining amount of* $3-\alpha$ *is split equally in expectation between airlines 2 and 3.*

These expected payoffs depend on the bargaining parameter α*. Airline 1 might do better or worse than its Shapley value payoff of* $2/3$*. This makes sense. If* $\alpha = 1$*, it is advantageous to be the first player at risk of elimination. If* $\alpha = 1/2$*, so that the eliminated player receives one half of its marginal contribution, then airline 1 receives* $1/2$*, which is less than its the Shapley value.*

For a fixed value of α, our weighted Shapley value yields a range of allocations to a given player, depending on the weights on different orders of risk of elimination. In Example 6, for $\alpha = 1/2$, airline 1's expected payoff lies in the interval $[1/2, 2/3]$. The best case for airline 1 is when the order of risk of elimination is random, in which case it pays $1/3$ of the cost of the first leg and therefore nets $2/3$. The worst case for airline 1 is when it is the first to be at risk of elimination. In this case, airlines 2 and 3 have, in effect, pre-formed an alliance. Airline 1 is bargaining with the coalition $\{2,3\}$, and therefore pays $1/2$ and nets $1/2$. (These calculations for airline 1 are independent of the length of runway required by airlines 2 and 3, provided the length exceeds 1.) Generalizing from this example, we believe that calculating an interval of potential payoffs based on varying weights, where the variation comes from different assumptions about risk of elimination, make intuitive sense.

Our weighted procedure can also be run in the forward direction, as in our α-procedure. Returning to Example 6, we now assume that the first coalition of airlines to form is the one that creates the largest value, namely, the coalition $\{2,3\}$. Airline 1 joins last. The fact that the ordering is the same in reverse and forward directions is a consequence of there being only three players in the game. With four or more players, the forward and reverse weighted procedures will generally differ. For example, in the forward direction, the two players that create the largest value are the first coalition to form. But one of the players in this coalition might have the lowest marginal contribution to the grand coalition and would therefore be the first to be at risk of elimination in the reverse procedure. In general, the player's payoffs depend both on the endogenous weights and on whether the procedure runs forward or in reverse.

Summing up, our approach to defining a generalized Shapley value involves introducing weights that depend on the players' marginal contributions and are therefore endogenous to the game. Our weights are anonymous—a player's power depends of their marginal contributions, not on their identity. The endogeneity also delivers a symmetry property. Consider two players who have the same standalone values. Our approach delivers an equal split in the two-player bargaining between these players—because their marginal contributions to each other are always equal in this case. (The two players may receive unequal allocations in larger sets.) We think preserving symmetry in this two-player case is natural and that power—at least, endogenous power—should arise from asymmetries across players. In our generalized Shapley scheme, asymmetries are understood in terms of differences in marginal contributions. We generate a range of outcomes, where the boundary cases are: (i) a fully deterministic ordering defined by increasing marginal contribution; and (ii) a completely random ordering. We view all outcomes in this range as reasonable and of interest, while outcomes outside this range seem less obviously natural.

8 Discussion

This section goes into more detail on some related literature, comments on proportional solutions, offers a non-cooperative implementation of our procedure, and concludes.

8.1 Related Work

We previously noted the connection to O'Neill (1982) and Aumann and Maschler (1985). When there are two players and the "cloth" is fully claimed, our axioms yield the same answer as the principle of the divided cloth. Aumann and Maschler extend the divided cloth principle to games with more than two players by imposing a consistency axiom which, as is usual in cooperative game theory, relates the solution of the overall game to solutions of subgames. Our α-procedure extends the two-player case differently—the Interval Equality Axiom and the Priority Axiom directly specify the division of value in games with any number of players. Our axioms possess a flavor of the principle of the divided cloth in that they dictate equal division, but only among those who have a claim to the contested amount. With different extensions, Aumann and Maschler (1985) and this paper naturally obtain different answers, namely, the nucleolus vs. the Shapley value.

The idea that the joining player's value would be divided up between the joining player and the coalition being joined goes back to Nowak and Radzik (1994), Ju, Borm and Rurs (2007), and Malawski (2013). The key difference between our approaches is that the earlier papers employ an equal division among the members being joined. Even so, the share going to the joiner can vary. When $\alpha = 1/|S|$, so that the shares to the joiner and the joined coalition are proportional to their relative sizes, the expected payoffs are known as the solidarity value; see Nowak and Radzik (1994). When α is a constant independent of S, the expected payoffs are an $\alpha : 1-\alpha$ weighted average of the Shapley value and the egalitarian solution, $v(N)/|N|$.[6]

Malawski (2013) defines a family of procedures for a game, and provides necessary and sufficient conditions for the payoffs to be the expected values under a procedure. In this framework, the allocation within $S\backslash i$ depends on the order in which the players join S. For example, the allocation could all go to the player of $S\backslash i$ who joins first or to the player who joins just before i. This might appear to be an unequal division. But, as Malawski recognizes, the random order of joining implies that the expected allocation to each player in $S\backslash i$ is always equal. Thus, the expected payoff under any procedure where a fixed share α goes to the joiner and $1-\alpha$ to $S\backslash i$ is always an $\alpha : 1-\alpha$ weighted average of the Shapley value and the egalitarian solution.

While our α-procedure builds on this earlier work, an important difference is that we allow the weights to be a function of the marginal contributions. For each interval k, our weights are constant across players, but only players whose marginal contribution is at or above m_k get to participate in this interval. For this reason, the weights are not independent of marginal contributions. This flexibility allows our solution to be invariant to the standard normalization in which $v(i) = 0$ for all $i \in N$.

We assume that only players of S share in the gains when player $j \in S$ joins $S\backslash j$. In Felsenthal and Machover (1996, 1997), Bernardi and Freixas (2018), and Malawski (2013), the gains are distributed between members in S and those in $N\backslash S$. For example, in Felsenthal and Machover (1996, 1997) and Bernardi and Freixas (2018) players arrive in a random order, and each arriving player randomly votes "yes" or "no". The "yes" voters join the prior yes voters and are given their marginal contributions to that set. The "no" voters are given their marginal contributions to the full set N excluding the prior no voters. In Malawski (2013), the marginal contribution of the joining player could be distributed equally to the players in $N\backslash S$. However, Malawski shows that a procedure where gains are shared with players in $N\backslash S$ is equivalent to a

[6]This is different from the solution where player i receives $v(i) + [v(N) - \sum_i v(i)]/|N|$, which is known as the center of imputation set (Driessen and Funaki, 1991) or the equal-surplus solution (Moulin, 2003). The reason for this difference is that the previously studied procedures are not invariant to the standard normalization when $\alpha < 1$. This is illustrated by a two-player game in which $v(1) = 0, v(2) = 10$, and $v(12) = 10$, so that player 1 is a dummy player. Consider the solution where all the marginal contribution goes to the player being joined. On average, each player receives the egalitarian solution $v(12)/2 = 5$. If we normalize the game so that $v(1) = v(2) = 0$, then normalized $v(12) = 0$. Now, player 2 receives 10 (from the normalization) and 0 from the game, for 10 in total, while player 1 receives 0.

reordered procedure in which gains are exclusively shared with those in S. This reordering property extends to our procedure as well.

Various combinatorial formulas yield the Shapley value. For example, Kleinberg and Weiss (1985) and Rothblum (1985) give each player an equal share of the total and then make adjustments based on how much value the coalitions containing a given player create relative to the average coalition. Rothblum (1988) provides an elegant formulation that brings the combinatorial formulas from Shapley (1953), Harsanyi (1963), and Kleinberg and Weiss (1985) into a unified framework. Our α-procedure is a novel implementation of the Shapley value. Analogously, these other computational formulas may suggest other new procedures that lead to the Shapley value.

8.2 Proportional Solutions

A potential alternative to our Interval Equality Axiom is proportional division. The total amount d is divided in proportion to each player's claim c_j. We continue to suppose that claims are marginal contributions and the amount to be divided when player i joins the coalition $S\backslash i$ is i's marginal contribution m_i. The amount that player $j \in S\backslash i$ receives, summed over all cases (including when j is the joining player) is

$$\frac{1}{|S|}\sum_{i\in S}\frac{m_j}{\sum_{k\in S} m_k}\times(1-\alpha)m_i. \tag{17}$$

Combining this with the separate amount αm_j that player j receives when it joins, the resulting payoff to j is:

$$\frac{1}{|S|}[\alpha m_j + (1-\alpha)\sum_{i\in S}\frac{m_j}{\sum_{k\in S} m_k}\times m_i] = \frac{m_j}{|S|}. \tag{18}$$

Proportional division—at least, in this form—leads directly to the Shapley value.

Under proportionality, the division of each interval (defined in partition 1) depends on all the players' marginal contributions. Under our α-procedure, each interval is divided up in a fixed manner, namely equally, among those who have a claim to that interval. This observation explains why the fact that proportional division leads to the Shapley value does not contradict our uniqueness result. In the conditions for Theorem 2, the weights for each interval are fixed, independent of marginal contributions.

An issue with proportional division, as we have defined it, is that the joining player is included in the division. The players being joined might exclude the joining player when dividing up the $1-\alpha$ share since the joiner is already getting an α-share.[7] This route suggests a different proportional solution, one that excludes the joining player. Here, when player i joins $S\backslash i$, player $j \in S\backslash i$ receives:

$$\frac{m_j}{\sum_{k\in S\backslash i} m_k}\times(1-\alpha)m_i. \tag{19}$$

This form of proportional division—which we call restricted proportional division—has not, to our knowledge, been treated in the literature. We examine it briefly in the context of our earlier Example 6, but leave a thorough investigation to future work.

Example 6 contd. *We examine restricted proportional division when $\alpha = 0$. For positive values of α, the solution will be a weighted average of the Shapley value ($\alpha = 1$) and the $\alpha = 0$ solution. Under restricted proportional division, airline 1 receives $0+1+2/3$ when it arrives first, $0+2/3$ when it arrives second, and 0 when it arrives third. The expected value is $7/9$, so that airline 1 pays $2/9$. This is less than the $1/3$ that airline 1 pays under the Shapley value. If we were to change the parameters and increase the runway length required by airlines 2 and 3, airline 1's payment would fall, converging to 0 in the limit.*

Proportional division does not mean that airline 1 pays a share of the total cost proportional to the length of the runway it uses. That kind of proportional division would lead airline 1 to pay 2/5 and would be detrimental to the "smaller" airline relative to the Shapley value. The same detrimental result would arise if airline 1's share of the total value was based on the marginal contributions to the grand coalition. In proportional division as expressed in Equation 17, the proportions are based on the contributions to each subgame. Airlines 1 and 2 have an equal claim when 2 joins 1 (or vice versa), since the marginal contribution to this pairing is 1 for both airlines. In the example of restricted proportional division above, airline 1 does better than under the Shapley value—and therefore the α-procedure, as well. Airline 2 has a larger claim than airline 1 when 3 joins $\{12\}$. Even so, airline 1 does better under restricted proportional division than under the α-procedure because it receives a smaller share of a larger amount. It gets 1/3 of 2 rather than 1/2 of 1.

[7] When $\alpha = 0$, this argument has less force, since then the joiner does not receive a separate fraction of the total.

8.3 Non-Cooperative Implementation

There is a longstanding program to find non-cooperative implementations of cooperative solution concepts. Gul (1989, 1999) initiated this program in the case of the Shapley value, with further progress made by Winter (1994), Hart and Mas-Colell (1996), Hart and Levy (1999), Serrano (2005, 2021), Ju, Borm and Rurs (2007), and McQuillin and Sugden (2016).

We can convert our reverse α-procedure into a non-cooperative game following closely the approach in Hart and Mas-Colell (1996) and Ju, Borm and Rurs (2007). The reverse α-procedure involves a division of value between the player at risk of exclusion and the other players. The same division can be obtained via a suitably defined non-cooperative game. A player i is selected to be at risk of exclusion. A player j (possibly i) is then selected to be the proposer and to make an ultimatum offer to the other players. Player j claims the full marginal contribution of player i and offers everyone else (itself included) their Shapley value in the game without i. In equilibrium, such proposals are always accepted. Each player's chance of being selected as the proposer when i is at risk is $\pi_j(S, i; \alpha)/m_i$, where $\pi_j(S, i; \alpha)$ is given by Equation 5 in Section 4. In particular, player i has an α chance of being the proposer and the remaining $1 - \alpha$ probability is allocated among the other players (except for the reversion case). When all players have an equal chance of being at risk of exclusion, this non-cooperative game and our reverse α-procedure are analogs, and the equilibrium payoffs are equal to the Shapley values. This game is close to the game in Hart and Mas-Colell (1996), with two main differences. First, before proposing, player j knows which player i is at risk. Second, the selection probabilities for the proposer depend on the marginal contributions.

8.4 Conclusion

We introduce a new bargaining procedure that generalizes the original Shapley (1953) procedure in sharing gains in an $\alpha : 1 - \alpha$ ratio between the player joining a coalition and the existing members of that coalition. Our distinctive contribution is that we allow for an unequal division among the existing members, one that reflects different bargaining powers. In the spirit of the principal of the divided cloth, we allocate value among existing members equally, subject to respecting the marginal-contribution bound of each player in the coalition. The unequal bounds (or claims) are what leads to the unequal divisions. Surprisingly, our α-procedure leads to the Shapley value.

We did not define the α-procedure with the goal of arriving at the Shapley value. Instead, we aimed for a procedure that is balanced in how value is distributed—both between joiner and joined, and among existing coalition members. This objective makes the α-procedure more complicated than the Shapley procedure, but we believe the underlying logic behind the Interval Equality and Priority Axioms is straightforward and easily interpretable. Our main result (Theorem 1 and Corollary 1.1) says that these two axioms yield the Shapley value.

We also establish that our α-procedure is unique among weighting procedures that lead to the Shapley value and satisfy the Priority Axiom and an additional Monotonicity Axiom. The α-procedure also suggests a new solution concept for cooperative games that differs from the traditional weighted Shapley value in allowing endogenous weights that depend on marginal contributions. This generalization of the Shapley value is motivated by the idea that some coalitions might be more likely to form than others, depending on the value they create.

We believe that the α-procedure—especially in the focal case where $\alpha = 1/2$ rather than the original Shapley case where $\alpha = 1$—adds an important justification to both descriptive and normative uses of the Shapley value as a cooperative game-theoretic solution concept.

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