

3 On the existence of a “complete” possibility structure*

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3.1 Introduction

We define *interactive possibility structures* for games and show that a *complete* such structure does not exist. Connections are made to the current investigation into the epistemic status of various game-theoretic solution concepts.

Fix a game between two players, Ann and Bob. A very basic idea in game theory is that of *common belief of rationality*. Ann and Bob are both rational, Ann believes Bob is rational, Bob believes Ann is rational, Ann believes Bob believes she (Ann) is rational, and so on indefinitely. Now, under the usual definition, a *rational* player is one who chooses a strategy that is optimal, given his or her belief about the other players' strategies. So, to talk about the rationality of Ann and Bob, we have to talk about what each believes (about the other's strategy). To talk about Ann's belief in Bob's rationality, we have to talk about what Ann believes about what Bob believes (about her strategy). And so on.

With these kinds of considerations in mind, it becomes natural to try to construct some sort of space of all possible beliefs, beliefs about beliefs, . . . , about a given game. But does such a space exist? There is good reason to ask the question. After all, a space of all beliefs might sound rather like the kinds of “sets of everything” that are well known to cause difficulties in mathematics. (Think of the paradoxes of naive set theory, such as Russell's Paradox.¹)

This note presents an impossibility result which says that, if defined in at least one way, a space of all beliefs cannot exist. The theorem should be thought of as a kind of background result to the literature, which includes various papers that do succeed in constructing spaces of all beliefs about a given game. (A partial list is Armbruster and Boge (1979); Boge and Eisele (1979); Mertens and Zamir (1985); Brandenburger and Dekel (1993); Heifetz (1993); Epstein and Wang (1996); and

* To the memory of Esther Aumann. I am indebted to Bob Aumann, Jerry Keisler, and Gus Stuart for many discussions bearing on this chapter. Pierpaolo Battigalli, Ken Corts, Amanda Friedenberg, Bob Gibbons, Rena Henderson, Harold Kuhn, Martin Meier, Salvatore Modica, Barry Nalebuff, Dov Samet, and Marciano Siniscalchi provided important input. The original version of this chapter was titled “On the existence of a ‘complete’ belief model” (03/20/98). Financial support from Harvard Business School and the Stern School of Business is gratefully acknowledged.

Battigalli and Siniscalchi (1999).) This note makes clear that these “positive” existence results must depend on suitably restricting what beliefs are allowed to be present.

The next two sections present the impossibility theorem. After that, we explain how this result relates to the ongoing epistemic program in game theory. We also elaborate on how our nonexistence result relates to the existence results of the literature.

3.2 Possibility structures

This section presents a formalism with which to talk about the beliefs of the players in a game, what the players believe about one another’s beliefs, etc. One piece of notation: Given a set X , let $\mathcal{N}(X)$ denote the set of all nonempty subsets of X .

Definition 1 Fix nonempty sets S^a and S^b . An (S^a, S^b) -based (interactive) possibility structure is a structure

$$\langle S^a, S^b, T^a, T^b, v^a, v^b \rangle$$

where T^a and T^b are nonempty sets, v^a is a map from T^a to $\mathcal{N}(S^b \times T^b)$, and v^b is a map from T^b to $\mathcal{N}(S^a \times T^a)$. Members of T^a or T^b are called types. The subset $v^a(t^a)$ is called the possibility set of type t^a of Ann, and similarly for Bob. Members of $S^a \times T^a \times S^b \times T^b$ are called states (of the world).

For an interpretation, fix a two-player strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$, where S^a, S^b are the strategy sets and π^a, π^b are the payoff functions of Ann and Bob, respectively. A particular possibility structure, together with a particular state (s^a, t^a, s^b, t^b) , is then a specification of each player’s strategy and type. Moreover, each type gives – via the associated possibility set – the strategy-type pairs of the other player that the first player considers possible.² In the literature, this is a fairly standard epistemic model, with the one difference being that here we formalize belief as possibility rather than as the more customary probability.³

A possibility structure may well have “holes” in it, in the sense that not every possibility set that Ann could have is actually present. That is, there may be nonempty subsets of $S^b \times T^b$ that are not associated with any type in T^a . The same may be true for Bob, of course. A special case, then, is when all possibility sets of both players are present.

Definition 2 Fix nonempty sets S^a and S^b , and an associated possibility structure

$$\langle S^a, S^b, T^a, T^b, v^a, v^b \rangle.$$

The structure is complete if v^a and v^b are onto.

In a complete structure, for every possibility set of Ann, there is a type of Ann with that set, and similarly for Bob.

The next section shows that, subject to a nontriviality condition, a complete structure does not exist. We discuss some implications of this result in Section 3.4.

3.3 The result

This section states and proves our nonexistence result.

Proposition 1 Fix nonempty sets S^a and S^b . Suppose that $|S^a| > 1$ or $|S^b| > 1$, or both. Then a complete (S^a, S^b) -based possibility structure does not exist.

To prove the result, we will use Cantor's Theorem in the following form.

Theorem 1 (Cantor) Fix a set X . If $|X| > 1$, then there is no onto map from X to $\mathcal{N}(X)$.

Proof Suppose, contra hypothesis, that there is such a map, to be denoted by d , and consider Cantor's diagonal set $D = \{x \in X: x \notin d(x)\}$. If D is nonempty, there is then a $y \in X$ such that $d(y) = D$, and the usual contradiction results. So suppose D is empty, that is, $x \in d(x)$ for each x in X . This implies that $d^{-1}(\{x\}) = \{x\}$ for each x in X . But then, using $|X| > 1$, it follows that there is no x in X such that $d(x) = X$, contradicting the assumption that d is onto.

Proof of Proposition 1 Suppose, contra hypothesis, that there is a complete structure

$$\langle S^a, S^b, T^a, T^b, v^a, v^b \rangle.$$

Step 1: By assumption, there is an onto map from T^a to $\mathcal{N}(S^b \times T^b)$.

Step 2: Map any $E \subseteq S^b \times T^b$ to the projection on T^b of E . This gives an onto map from $\mathcal{N}(S^b \times T^b)$ to $\mathcal{N}(T^b)$.

Step 3: There is a 1 to 1 map from T^b to $\mathcal{N}(T^b)$ that maps t^b to $\{t^b\}$. Thus, there is an onto map from $\mathcal{N}(T^b)$ to T^b .

Step 4: By assumption, there is an onto map from T^b to $\mathcal{N}(S^a \times T^a)$.

Step 5: Just as in Step 2, there is an onto map from $\mathcal{N}(S^a \times T^a)$ to $\mathcal{N}(T^a)$.

Putting steps 1 through 5 together yields an onto map from T^a to $\mathcal{N}(T^a)$. Now suppose that $|S^b| > 1$, and fix $r^b, s^b \in S^b$ with $r^b \neq s^b$. Then $\mathcal{N}(S^b \times T^b)$ contains the distinct elements $\{r^b\} \times T^b$ and $\{s^b\} \times T^b$, and so, by step 1, we certainly have $|T^a| > 1$. This now contradicts Cantor's Theorem. The case $|S^a| > 1$ is treated similarly.

3.4 Discussion

In contrast to our nonexistence result, Mariotti and Piccione (1999) and Meier (2001) show that a complete possibility structure does exist if the underlying spaces S^a, S^b are compact Hausdorff and possibility sets are required to be (nonempty) compact.⁴ Salonen (1999) gives a variety of positive results on completeness, under a variety of structural assumptions.

How do these existence results fit with our result? The answer, of course, is that whether or not a possibility structure containing all possibility sets exists depends crucially on just how the term "all possibility sets" is understood. Our

impossibility theorem should be thought of as a kind of baseline result, which says that completeness is impossible if literally all possibility sets are wanted. But if we make topological assumptions that serve to rule out certain kinds of possibility sets, then a (restrictedly) complete structure may exist.

There is also the comparison between our nonexistence result and the standard existence result of Mertens and Zamir (1985) and others.⁵ The results of Mariotti–Piccione, Meier, and Salonen make clear that the issue here is not our use of possibility versus probability, as in Mertens–Zamir. Rather, it is whether or not certain topological assumptions are made.

Next, there is the connection to the epistemic program in game theory. The connection is that two recent papers in this literature, namely Battigalli and Siniscalchi (2002) and Brandenburger and Keisler (2000), use complete structures. Battigalli–Siniscalchi give an epistemic characterization of extensive-form rationalizability (Pearce 1984), while Brandenburger–Keisler provide epistemic conditions for iterated admissibility (iterated weak dominance). Both papers formalize belief as probability and also make various topological assumptions to get the complete structures they need. (As above, though, the use of probabilities is less critical than are the topological assumptions.) Arguably, this chapter shows that there is a basic impossibility underneath the Battigalli–Siniscalchi and Brandenburger–Keisler analyses: completeness is impossible unless the beliefs that the players can have are suitably restricted.

Finally, Brandenburger and Keisler (1999) give a model-theoretic impossibility result that is much stronger than the one here. Given any possibility structure, they define a naturally associated first-order language. They then show that no possibility structure contains every possibility set that is definable in this language. In short, no structure is *definably complete*.⁶

This result offers a more basic way to talk about the boundary between existence and nonexistence of complete structures. We can now think of completeness as relative to a language. We have to say how the players think before we can say whether everything they can think of is present. This brings us to a current area of research: give model-theoretic treatments of the various positive results on completeness that we mentioned above. The point is to make explicit the kinds of reasoning the players can and cannot be allowed to do if completeness is to be possible.⁷ And from this, we can hope to get a purely model-theoretic – or “logical” – analysis of various game-theoretic solution concepts. This will be a deeper understanding than we have now.

Notes

- 1 To remind the reader, Russell’s Paradox concerns “the collection of all sets which are not members of themselves.” The contradiction arises if this collection is a set since then it is a member of itself if and only if it is not a member of itself.
- 2 Under another interpretation, which allows for uncertainty about both the “structure” and the “conduct” of the game, the set S^a could be the product of Ann’s strategy set and a set of payoff functions for Ann. (Likewise for Bob.) Relatedly, the sets S^a and S^b in Definition 1 need not be finite or have any particular structure.
- 3 We return to the possibility versus probability distinction in Section 3.4.

- 4 Meier (2001) covers so-called *conditional* possibility structures, which include (ordinary) possibility structures as a special case.
- 5 As referenced in the Introduction. Mertens–Zamir and the other papers show the existence of a so-called *universal* structure, but the existence of a complete structure is an immediate corollary.
- 6 Other related impossibility results include Gilboa (1988), Fagin *et al.* (1991), Fagin (1994), Heifetz and Samet (1998), and Fagin *et al.* (1999).
- 7 Meier (2001) gives such a treatment.

References

- Armbruster, W. and W. Boge, "Bayesian Game Theory," in O. Moeschlin and D. Pallaschke (eds), *Game Theory and Related Topics*, North-Holland, Amsterdam, 1979.
- Battigalli, P. and M. Siniscalchi, "Hierarchies of Conditional Beliefs and Interactive Epistemology in Dynamic Games," *J. Econ. Theory*, 88, 1999, 188–230.
- Battigalli, P. and M. Siniscalchi, "Strong Belief and Forward-Induction Reasoning," *J. Econ. Theory*, 106, 2002, 356–391.
- Boge, W. and T. Eisele, "On Solutions of Bayesian Games," *Int. J. Game Theory*, 8, 1979, 193–215.
- Brandenburger, A. and E. Dekel, "Hierarchies of Beliefs and Common Knowledge," *J. Econ. Theory*, 59, 1993, 189–198.
- Brandenburger, A. and H.J. Keisler, "An Impossibility Theorem on Beliefs in Games," unpublished, 1999.
- Brandenburger, A. and H.J. Keisler, "Epistemic Conditions for Iterated Admissibility," unpublished, 2000.
- Epstein, L. and T. Wang, "Beliefs about Beliefs Without Probabilities," *Econometrica*, 64, 1996, 1343–1374.
- Fagin, R., "A Quantitative Analysis of Modal Logic," *J. Symbolic Logic*, 59, 1994, 209–252.
- Fagin, R., J. Geanakoplos, J. Halpern, and M. Vardi, "The Hierarchical Approach to Modeling Knowledge and Common Knowledge," *Int. J. Game Theory*, 28, 1999, 331–365.
- Fagin, R., J. Halpern, and M. Vardi, "A Model-Theoretic Analysis of Knowledge," *J. Assn. Computing Machinery*, 38, 1991, 382–428.
- Gilboa, I., "Information and Meta Information," in M. Vardi (ed.), *Proceedings of the Second Conference on Theoretical Aspects of Reasoning about Knowledge*, Morgan Kaufmann, 1988, 227–243.
- Heifetz, A., "The Bayesian Formulation of Incomplete Information – The Non-Compact Case," *Int. J. Game Theory*, 21, 1993, 329–338.
- Heifetz, A. and D. Samet, "Knowledge Spaces with Arbitrarily High Rank," *Games Econ. Behavior*, 22, 1998, 260–273.
- Mariotti, T. and M. Piccione, "A Hierarchical Construction of a Compact Possibility Model," unpublished, London School of Economics, 1999.
- Meier, M., "Conditional Possibility Structures," unpublished, Universities of Bielefeld and Caen, 2001.
- Mertens, J-F. and S. Zamir, "Formulation of Bayesian Analysis for Games with Incomplete Information," *Int. J. Game Theory*, 14, 1985, 1–29.
- Pearce, D., "Rational Strategic Behavior and the Problem of Perfection," *Econometrica*, 52, 1984, 1029–1050.
- Salonen, H., "Beliefs, Filters, and Measurability," unpublished, University of Turku, 1999.