Correlation in Games:

A Case Study in Epistemic Game Theory

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References

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"The Power of Paradox: Some Recent Developments in Interactive Epistemology," *International Journal of Game Theory*, 35, 2007, 465-492.

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A Puzzle: Correlation in Games

X

	L	*R*
U	3	3
D	0	0

Y

	L	*R*
U	2	0
D	0	2

Z

	L	*R*
U	0	0
D	3	3

Strategy *Y* is **undominated** --- i.e., there is no other (mixed) strategy that always does better.

It is therefore optimal under some probability measure --- e.g., under

$$\mathrm{Prob}(U, L) = \mathrm{Prob}(D, R) = 1/2$$

But, there is no product measure under which *Y* is optimal.

[Let *p* be the probability on *U* and *q* be the probability on *L*. Then:

$$\max\{3p, 3(1-p)\} > 2pq + 2(1-p)(1-q) \quad]$$

Where Does the Correlation Come From?

Where does the correlation come from?

In **cooperative** game theory, players can choose strategies jointly --- we will come back to this later.

But, in **non-cooperative** game theory, players choose separately (“locally”).

In non-cooperative theory, we need to look for extra (“hidden”) variables to create correlation.

Complete or Incomplete Description?

In classical game theory, the matrix (or tree) is the **complete** definition of a game.

Therefore, we have to look beyond the game for extra variables (usually called **signals**).

In **epistemic game theory** (**EGT**), the matrix (or tree) is an **incomplete** definition.

EGT includes in the definition of a game the concept of an **epistemic type** for a player.

A type for a player describes what the player thinks about what strategies the other players choose, about what the other players think about this, ….

What does game theory look like if we create correlations through types?

An Epistemic Game

X

	L	R
U	1,1,3	1,0,3
D	0,1,0	0,0,0

Y

	L	R
U	1,1,2	0,0,0
D	0,0,0	1,1,2

Z

	L	R
U	1,1,0	1,0,0
D	0,1,3	0,0,3

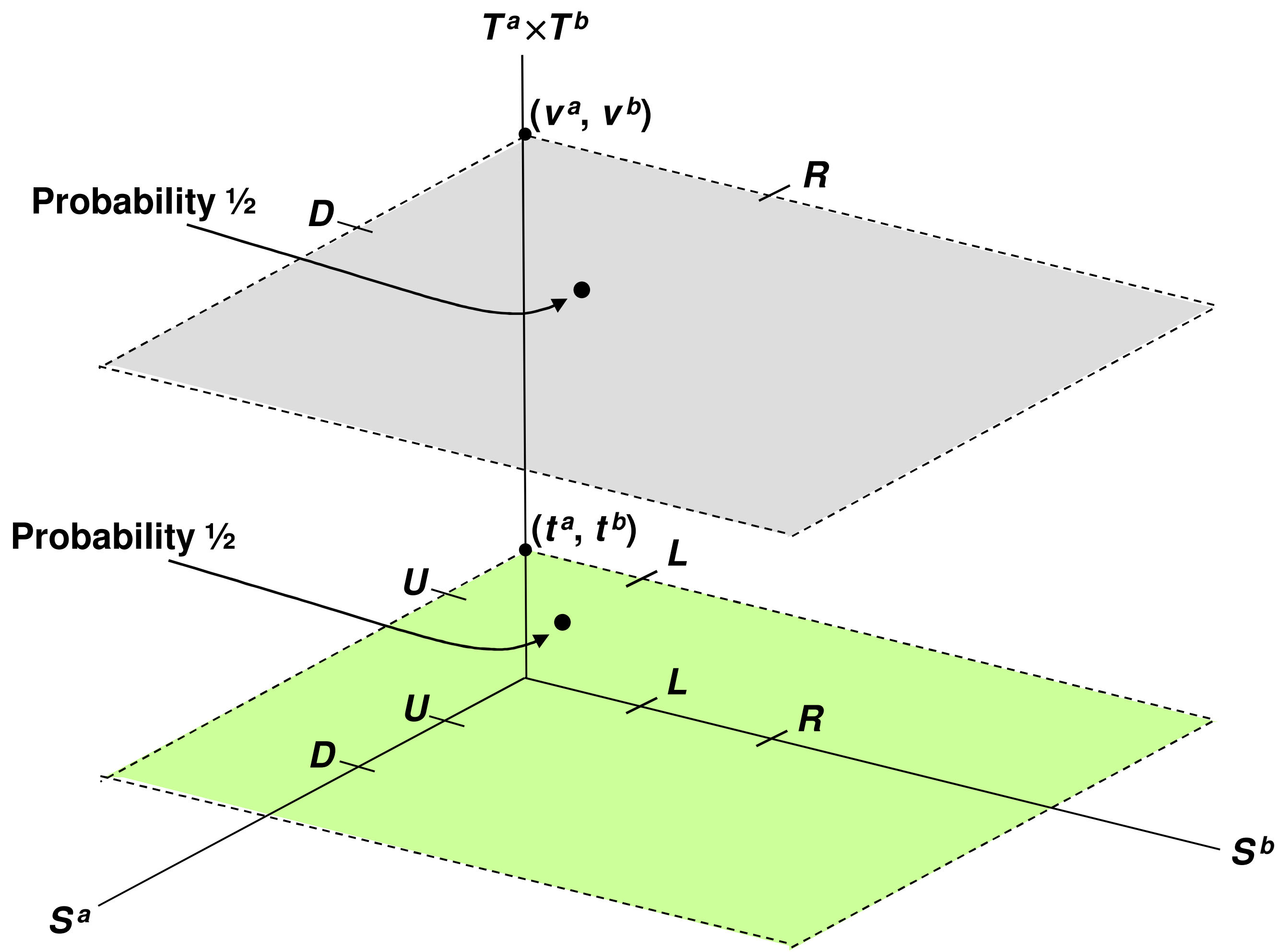
Suppose:

Ann's strategy set $S^a = \{U, D\}$ Ann's type set $T^a = \{t^a, v^a\}$

Bob's strategy set $S^b = \{L, R\}$ Bob's type set $T^b = \{t^b, v^b\}$

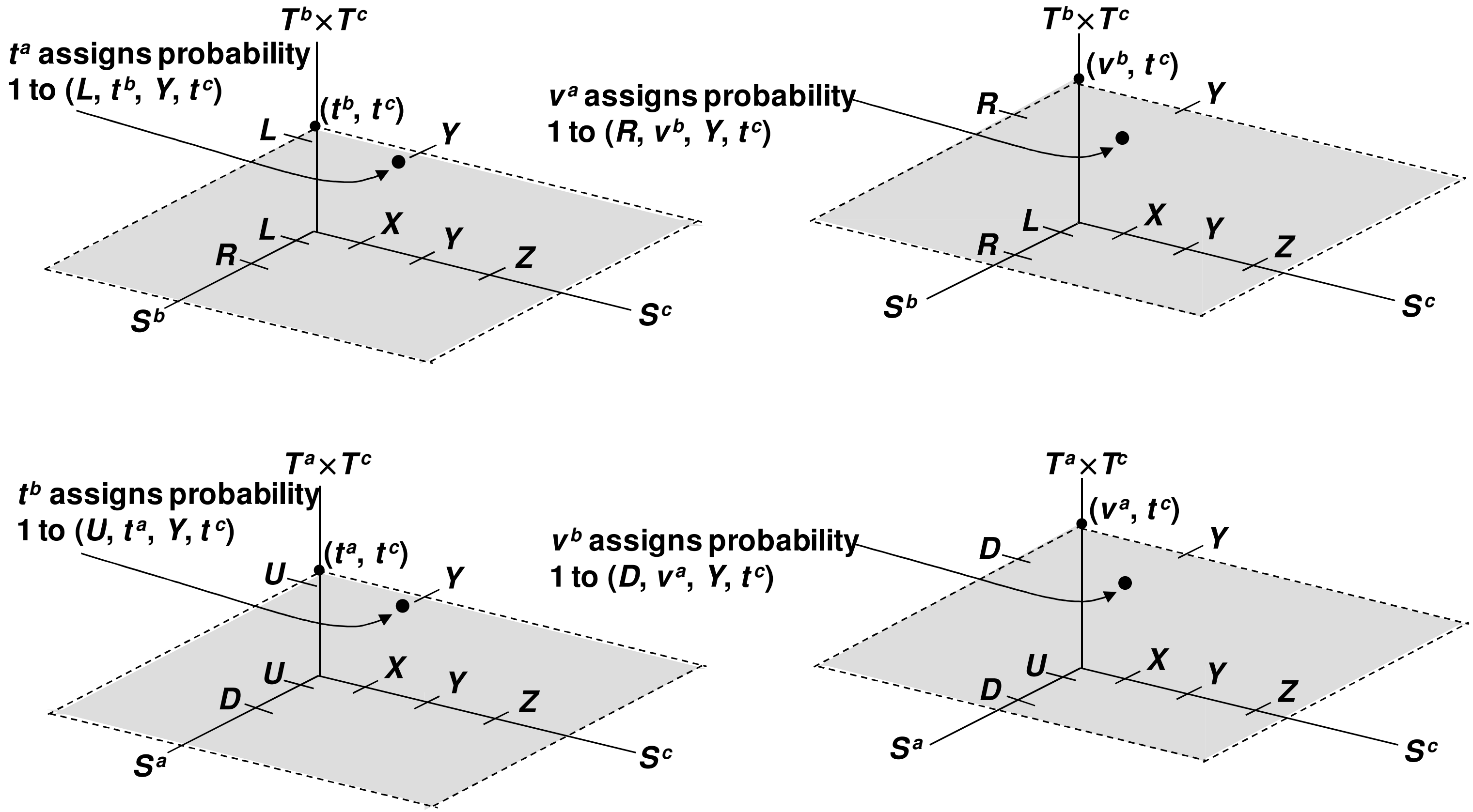
Charlie's strategy set $S^c = \{X, Y, Z\}$ Charlie's type set $T^c = \{t^c\}$

A Type Structure



Type t^c assigns probabilities 1/2 to (U, t^a, L, t^b) and 1/2 to (D, v^a, R, v^b)

Type Structure Cont’d



Conditions on Type Structures

Definition (**Conditional Independence**, **CI**): *Charlie's type* t^c *should satisfy*

$$\mathrm{Prob}_{t^c}(s^a, s^b | t^a, t^b) = \mathrm{Prob}_{t^c}(s^a | t^a, t^b) \times \mathrm{Prob}_{t^c}(s^b | t^a, t^b).$$

Definition (**Sufficiency**, **SUFF**): *Charlie's type* t^c *should satisfy*

$$\begin{aligned} \mathrm{Prob}_{t^c}(s^a | t^a, t^b) &= \mathrm{Prob}_{t^c}(s^a | t^a), \\ \mathrm{Prob}_{t^c}(s^b | t^a, t^b) &= \mathrm{Prob}_{t^c}(s^b | t^b). \end{aligned}$$

… and analogous conditions on Ann and Bob.

Technical notes:

(i) If there are redundant types, then the conditioning must be on hierarchies.

(ii) The definitions can be extended to infinite type spaces.

No "Physical" Correlation

Lemma: *Under CI and SUFF, if*

$$\mathrm{Prob}_{t^c}(t^a, t^b) = \mathrm{Prob}_{t^c}(t^a) \times \mathrm{Prob}_{t^c}(t^b)$$

then

$$\mathrm{Prob}_{t^c}(s^a, s^b) = \mathrm{Prob}_{t^c}(s^a) \times \mathrm{Prob}_{t^c}(s^b).$$

A correlated assessment about strategy choices implies a correlated assessment about types (more precisely: hierarchies of beliefs).

This is the expression of no joint choices of strategies --- or, no "physical" correlation.

(The conditions are tight.)

Why Correlated Types: An Analogy

We can imagine a game before the game …

"Penrose's proposal has another advantage, in common with other hypotheses that eliminate the singularity. It suggests that before the Big Bang, there would have been plenty of time to set up the correlations seen in observations of the cosmic microwave background and distributions of galaxies."

-- Lee Smolin, review of *Cycles of Time: An Extraordinary New View of the Universe*, by Roger Penrose (Bodley Head, 2010); in *Nature*, 467, 10/28/10, 1034–1035.

My thanks to Rohit Parikh for suggesting this analogy.

The Question Formalized

Theorem ("The Fundamental Theorem of EGT"): *The epistemic condition of **rationality and common belief of rationality** (**RCBR**) is characterized by **iterated strong dominance**.*

Question: What is the effect of imposing CI and SUFF in addition?

Notes:

(i) There is an equivalence between subjective correlated equilibrium (Aumann 1974) and iterated strong dominance. (See Brandenburger and Dekel 1987.)

(ii) See later for the relationship to objective correlated equilibrium.

A Second Game

	L	C	R
U	0,0,2	0,0,2	0,1,2
M	0,0,0	0,0,0	0,1,0
D	1,0,2	1,0,2	1,1,2

X

	L	C	R
U	1,1,1	0,1,0	0,1,0
M	1,0,0	0,0,1	0,1,0
D	1,0,0	1,0,0	1,1,0

Y

	L	C	R
U	0,0,0	0,0,0	0,1,0
M	0,0,2	0,0,2	0,1,2
D	1,0,2	1,0,2	1,1,2

Z

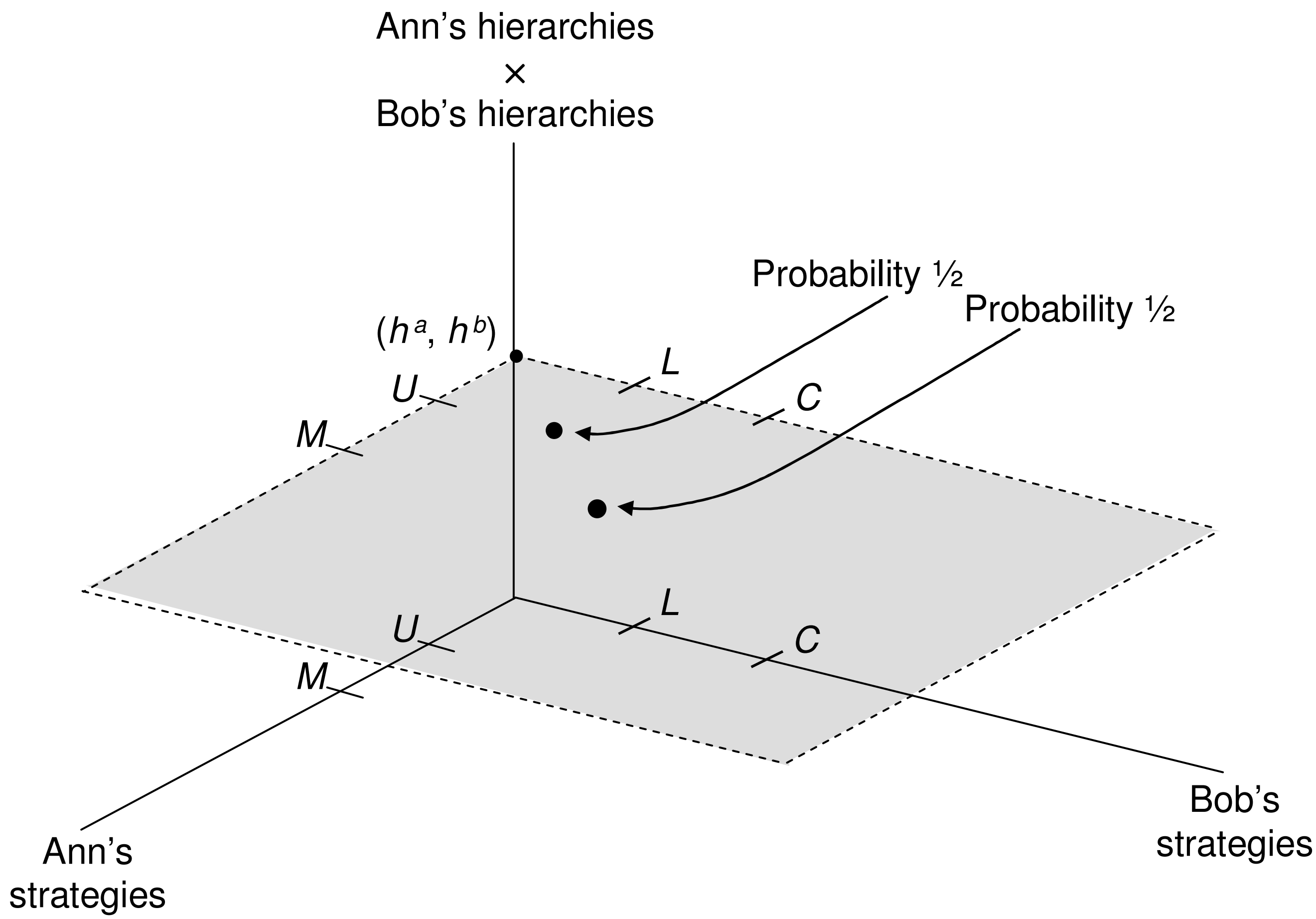
U and *M* are each optimal if and only if $\mathrm{Prob}(L, Y) = 1$

L and *C* are each optimal if and only if $\mathrm{Prob}(U, Y) = 1$

Y is optimal if and only if $\mathrm{Prob}(U, L) = \mathrm{Prob}(M, C) = 1/2$

Every strategy survives iterated strong dominance.

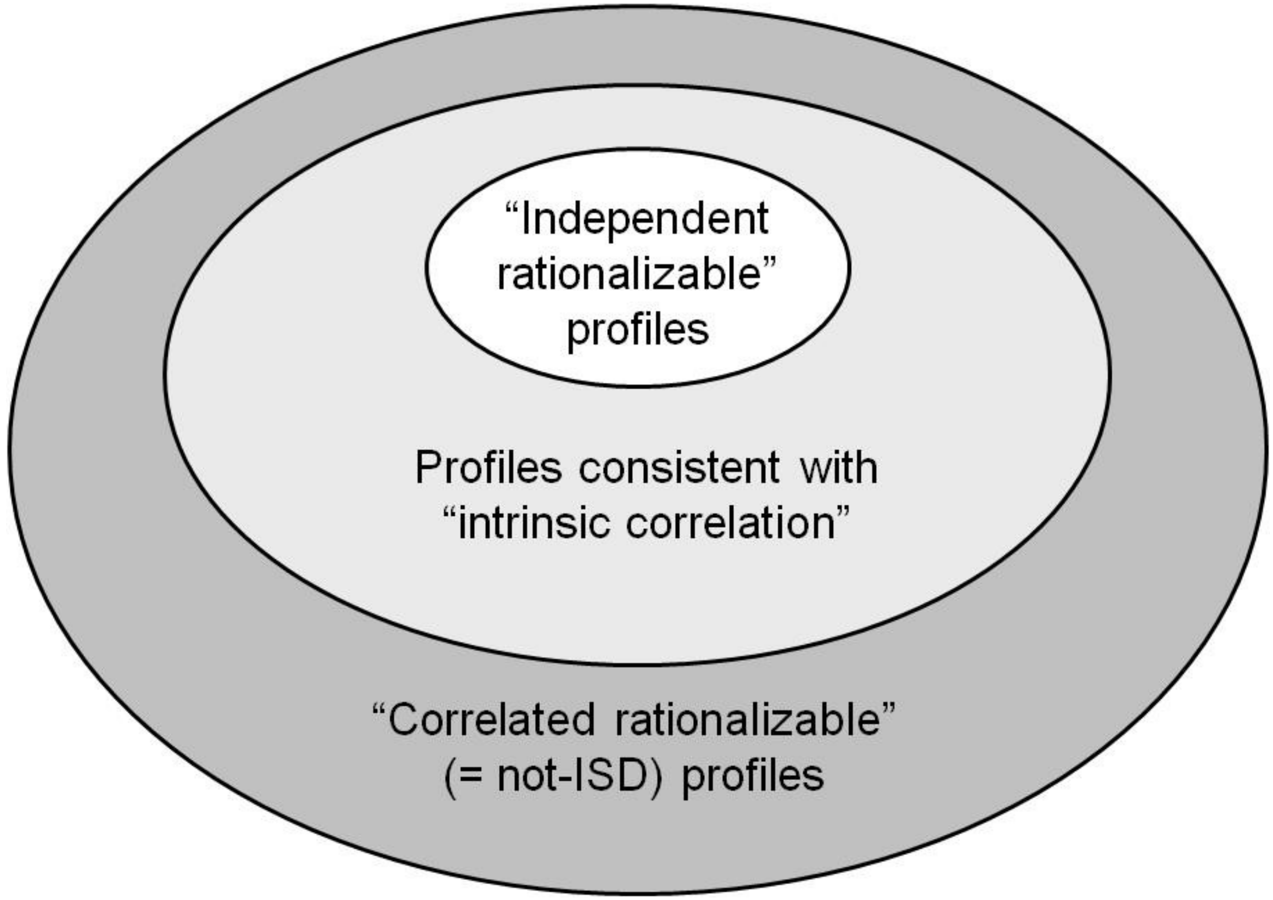
A Second Game Cont'd



Consider a strategy-type pair (Y, t^c) for Charlie that is rational, believes Ann and Bob are rational, believes ….

But, CI requires that Charlie's conditional probability measure, conditional on any horizontal plane, must be a product measure --- contradiction!

Summary of Relationships



Resolution? "Intrinsic" Correlation

1. We can do EGT (under CI and SUFF).

2. Can we provide a characterization of RCBR, CI, and SUFF in terms of the strategies that can be played?

   -- Du (2008) and Peysakhovich (2009) give partial answers

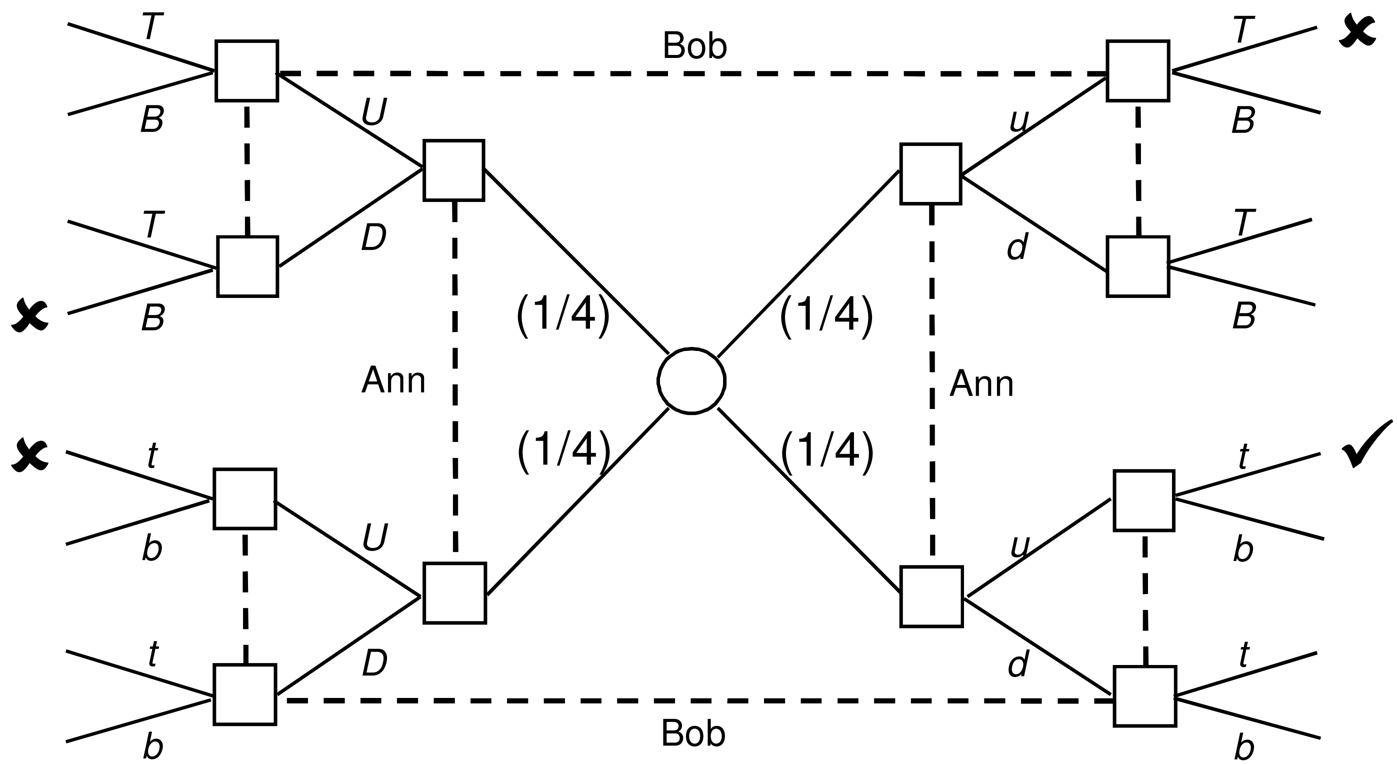
3. How does our analysis extend to the tree (cf. Kohlberg and Reny 1997 on independence in conditional probability systems)?

Resolution? "Extrinsic" Correlation

1. We can add extrinsic signals to the game (Aumann 1974).

   -- This means we are no longer analyzing the original game.

2. What dependency should we allow between (payoff-relevant) chance moves inside the game and (payoff-irrelevant) signals outside the game?



Cf. "Nonlocality for Two Particles Without Inequalities for Almost All Entangled States," by Lucien Hardy, *Physical Review Letters*, 71, 1993, 1665-1668.

Resolution? "Physical" Correlation

1. In the second game, we can allow Charlie to think that Ann and Bob jointly choose (U, L) or jointly choose (M, C).

   -- Now, the analyst thinks that each player chooses separately, yet the analyst simultaneously thinks that a player thinks otherwise!

2. Should we (the analysts) instead suppose that all subsets of players can choose strategies jointly?

   -- This route appears to blur (erase?) the boundary between non-cooperative and cooperative theory!

Historical note:

Cooperative games were introduced by von Neumann (1928) as arising from coordinated behavior in non-cooperative games.