Agreement and Disagreement in a Non-Classical World

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“We are all agreed that your theory is crazy. The question that divides us is whether it is crazy enough to have a chance of being correct.”

— Niels Bohr
The Classical Agreement Theorem

Alice and Bob possess a common prior probability distribution on a state space

They each then receive different private information about the true state

They form their conditional (posterior) probabilities \( q_A \) and \( q_B \) of an underlying event of interest

Theorem (Aumann, 1976): *If these two values \( q_A \) and \( q_B \) are common knowledge between Alice and Bob, they must be equal*

Here, an event \( E \) is **common knowledge** between Alice and Bob if they both know it, both know they both know it, and so on indefinitely.
Applications

The agreement theorem is considered a basic requirement in classical epistemics. It has been used to:

- show that two risk-neutral agents, starting from a common prior, cannot agree to bet with each other (Sebenius and Geanakoplos, 1983)
- prove “no-trade” theorems for efficient markets (Milgrom and Stokey, 1982)
- establish epistemic conditions for Nash equilibrium (Aumann and Brandenburger, 1995)

The Role of Common Knowledge: A “Discontinuity” at Infinity

J. Geanakoplos and H. Polemarchakis, “We Can’t Disagree Forever,” Journal of Economic Theory, 28, 1982, 192–200; this variant is due to John Geanakoplos (private communication)
Non-Classical Settings I

What is the status of the Agreement Theorem when classical probability theory does not apply?

In the physical domain, the canonical case is quantum mechanics, where a fundamental result (Bell’s Theorem, 1964) says that no “local hidden-variable” theory can model the results of all quantum experiments.

As we will see, this implies that the classical Bayesian model does not apply.

In probability theory, there is a finite analog to the de Finetti representation theorem for infinite sequences of exchangeable random variables, if mixing is via a signed probability measure (Jaynes, 1986; Kearns and Székely, 2006; Janson, Konstantopoulos, and Yuan, 2016).

This permits an exchangeability derivation of Fermi-Dirac statistics, paralleling an infinite exchangeability derivation of Bose-Einstein statistics (Kearns and Székely, 2006; Bach, Blank, and Francke, 1985).

Non-Classical Settings II

In decision theory, Perea (2022) axiomatizes expected utility theory for conditional preference relations

A conditional preference relation assigns to every possible probabilistic belief on a (finite) set of states, a preference relation over the decision maker’s (finite) set of actions

The motivation is that, in a game, we typically fix a player’s utility function but not their beliefs — what, then, does the utility function represent?

The interest is in axiomatizing ordinary expected utility (for unsigned probability measures)

But, to obtain sufficient bite for the axioms introduced, the analysis must include scenarios where the decision maker holds a signed probability measure on the states
**Quantum Theory: 2 x 2 x 2 Boxes**

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**Empirical model:**

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**Bell model:**
### Phase-Space Representation

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An Impossibility Result

From phase space and the Bell empirical model, we can calculate

\[ p_0 + p_1 + p_4 + p_5 = 1/2 \]
\[ p_4 + p_5 + p_{12} + p_{13} = 1/8 \]
\[ p_1 + p_3 + p_5 + p_7 = 1/8 \]
\[ p_0 + p_2 + p_8 + p_{10} = 1/8 \]

Adding the second, third, and fourth equations gives

\[ p_0 + p_1 + p_2 + p_3 + p_4 + 2p_5 + p_7 + p_8 + p_{10} + p_{12} + p_{13} = 3/8 \]

which contradicts the first equation

Theorem (Abramsky and Brandenburger, 2011): An empirical model is “no signaling” if and only if there is a phase-space model with a signed probability measure that induces it

General Set-up

There is a finite abstract state space $\Omega$

Alice and Bob have partitions $\mathcal{P}_A$ and $\mathcal{P}_B$ of $\Omega$ representing their private information

There is a common – possibly signed – prior probability measure $p$ on $\Omega$

Observability:

Assume throughout that all members of the partitions $\mathcal{P}_A$ and $\mathcal{P}_B$ receive probability in the interval $(0,1]$ 

Assume, too, that all events of interest receive probability in $(0,1]$
A Warm-Up Example

Alice’s (Bob’s) partition is red (blue)

The event of interest is
\[ E = \{ \omega_1, \omega_3, \omega_4 \} \]

The true state is \( \omega_1 \)

At \( \omega_1 \), Alice assigns (conditional) probability 1 to \( E \)

At \( \omega_1 \), Bob assigns (conditional) probability 0 to \( E \)

The event that Bob assigns probability 0 to \( E \) is
\[ F = \{ \omega_1, \omega_2, \omega_3 \} \]

At \( \omega_1 \), Alice assigns probability 1 to \( F \)

Call this singular disagreement

It is impossible classically!
From Knowledge to Certainty

Definition: Alice knows an event $E$ at state $\omega$ if $\mathcal{P}_A(\omega) \subseteq E$

Definition: Alice is certain of an event $E$ at a state $\omega$ if $p(E \mid \mathcal{P}_A(\omega)) = 1$

Fix an event $E$ and probabilities $q_A$ and $q_B$, and let

$$A_0 = \{ \omega \in \Omega : p(E \mid \mathcal{P}_A(\omega)) = q_A \}$$

$$B_0 = \{ \omega \in \Omega : p(E \mid \mathcal{P}_B(\omega)) = q_B \}$$

$$A_{n+1} = A_n \cap \{ \omega \in \Omega : p(B_n \mid \mathcal{P}_A(\omega)) = 1 \}$$

$$B_{n+1} = B_n \cap \{ \omega \in \Omega : p(A_n \mid \mathcal{P}_B(\omega)) = 1 \}$$

for all $n \geq 0$

Definition: It is common certainty at a state $\omega^*$ that Alice assigns probability $q_A$ to $E$ and Bob assigns probability $q_B$ to $E$ if $\omega^* \in \bigcap_{n=0}^\infty A_n \cap \bigcap_{n=0}^\infty B_n$
Relationship Between Knowledge and Certainty

If Alice knows an event $E$ at state $\omega$, then she is certain of $E$ at $\omega$

It is also true that common knowledge of $E$ implies common certainty of $E$

(Proof: If Alice knows Bob knows $E$, then she knows Bob is certain of $E$, since knowledge is monotonic. From this, Alice is certain Bob is certain of $E$. The argument continues to higher levels.)

Arguably, the distinction between these modalities is “small” in the classical domain (arguably, not!)

Also, in the classical domain, there is an Agreement Theorem for common certainty

Theorem (classical): Fix a (non-negative) common prior and an event $E$. Suppose at a state $\omega^*$ it is common certainty that Alice’s probability of $E$ is $q_A$ and Bob’s probability of $E$ is $q_B$. Then $q_A = q_B$.

But what happens in the non-classical world?
Non-Classical Agreement with Knowledge

Even without our observability conditions, we get a non-classical analog to the classical Agreement Theorem

Theorem (non-classical): Fix a signed common prior and an event $E$. Suppose at a state $\omega^*$ it is common knowledge that Alice’s probability of $E$ is $q_A$ and Bob’s probability of $E$ is $q_B$. Then $q_A = q_B$.

Proof: Follow closely the classical argument. Consider the (equal) conditional probabilities $q_A$ for Alice, calculated for members of her partition that are contained in the member of the meet $(\mathcal{P}_A \land \mathcal{P}_B)(\omega^*)$. This time, we take an affine rather than convex combination of this constant probability to get $p(E \mid (\mathcal{P}_A \land \mathcal{P}_B)(\omega^*)) = q_A$. Then run the same argument for Bob.

But let’s see what happens with common certainty ...

Common Certainty of Disagreement

The event of interest is

\[ E = \{ \omega_2, \omega_4, \omega_5, \omega_6 \} \]

The true state is \( \omega_5 \)

At \( \omega_5 \), it is common certainty that Alice assigns probability \( 1 - 2\epsilon \) to \( E \) while Bob assigns probability \( 1 - 2\eta \) to \( E \)

That is, there is common certainty of disagreement!
Alice announces a probability of $1 - 2\epsilon$ (à la Geanakoplos and Polemarchakis, 1982)

Bob infers that Alice observed $\{\omega_1, \omega_2, \omega_5\}$ and announces a probability of 1

Alice needs to ask what Bob would have inferred from her announcement had he observed $\{\omega_1, \omega_2, \omega_6\} - \text{but this would have led Bob to announce } -\epsilon/0!$
Communication II

This heuristic treatment exposes two issues:

1. We saw the emergence of an ill-defined conditional probability

2. We ignored the fact that Bob assigns probability 0 to Alice’s having the information \( \{\omega_3, \omega_4, \omega_6\} \)

Concerning 2., we implicitly assumed that Bob updates by putting probability 1 on this event

We will tackle the first issue but not the second

To tackle the second issue, we would need an extension of the concept of a conditional probability system (Rényi, 1955) to signed probabilities — which appears to be an open direction

But note again that the preceding treatment is only heuristic motivation for what follows

Communication-Enabled Structures

Define a sequence of partitions for Alice, corresponding to announcements she could make about her probability of $E$, her certainty of Bob’s probability, etc., and likewise for Bob

$$\mathcal{M}_{A}^{(n)} = \{A_n, A_n^c\}$$

$$\mathcal{M}_{B}^{(n)} = \{B_n, B_n^c\}$$

For any $\pi, E \subseteq \Omega$, say $\pi$ is regular with respect to $E$ if $p(\pi) \geq 0$ and $0 \leq p(\pi \cap E) \leq p(\pi)$

A structure $(\Omega, p, \mathcal{P}_A, \mathcal{P}_B)$ is communication-enabled with respect to $E$ if for each $n \geq 0$, each $\pi \in \mathcal{P}_A \lor \mathcal{M}_B^{(n)}$ and each $\pi \in \mathcal{P}_B \lor \mathcal{M}_A^{(n)}$ is regular with respect to $E$

Note: This property fails in the previous example
A New Agreement Theorem

Theorem: Fix a structure that is communication-enabled with respect to $E$ and suppose at a state $\omega^*$ it is common certainty that Alice’s probability of $E$ is $q_A$ and Bob’s probability of $E$ is $q_B$. Then $q_A = q_B$.

Notice that Alice’s potential announcements are made relative to her (initial) partition $\mathcal{P}_A$; and likewise for Bob

This is different from Geanakopolos and Polemarchakis (1982) where the (actual) announcements are made relative to updated partitions

Alternatively put, the mere ability to “confirm” the epistemic state (here, the state is common certainty of the posteriors) is enough to rule out disagreement – the confirmation need not actually be carried out
Realizability of Common Certainty of Disagreement?

In the physical domain, it can be shown that common certainty of disagreement (CCD) is impossible when observing quantum systems but possible for “superquantum” (no-signaling) systems.

The impossibility of CCD can therefore be proposed as a physical axiom.

In decision theory, if we equip agents with signed probability measures, it seems we can get highly non-classical behavior, such as betting between risk-neutral agents.

Or, should the impossibility of CCD be elevated to an (epistemic) decision-theoretic principle?

If yes, what non-classical behavior is then allowed? This appears to be an open direction ...
Two Alternative Models

1. Khrennikov and Basieva (2014) and Khrennikov (2015) consider quantum-like observers of a quantum system who employ either the knowledge or certainty modality.
   This approach allows CCD even for quantum systems.

2. (With thanks to Miklós Pintér) We could strengthen the belief modality to say:
   Alice is fully certain of $E$ if all events in the complement of $E$ receive probability 0.
   We could investigate this avenue by developing a preference-based definition of certainty (analogous to defining Savage-null events) from a decision theory with signed probabilities.
   This appears to be an interesting open direction.

Thank You