Agreement and Disagreement in a Non-Classical World

Adam Brandenburger, Patricia Contreras-Tejada, Pierfrancesco La Mura, Giannicola Scarpa, and Kai Steverson

New York University, Instituto de Ciencias Matemáticas, Madrid, HHL Leipzig Graduate School of Management, Universidad Complutense de Madrid, New York University

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The Classical Agreement Theorem

Alice and Bob possess a common prior probability distribution on a state space

They each then receive different private information about the true state

They form their conditional (posterior) probabilities q_A and q_B of an underlying event of interest

Theorem (Aumann [1976]): *If these two values* q_A *and* q_B *are common knowledge between Alice and Bob, they must be equal*

Here, an event E is common knowledge between Alice and Bob if they both know it, both know they both know it, and so on indefinitely

It is insufficient to assume that Alice and Bob have high-order mutual knowledge of the probabilities (Geanakoplos and Polemarchakis [1982], Aumann and Brandenburger [1995])

R. Aumann, “Agreeing to Disagree,” *Annals of Statistics*, 4, 1976, 1236-1239;
J. Geanakoplos and H. Polemarchakis, “We Can’t Disagree Forever,” *Journal of Economic Theory*, 28, 1982, 192-200;
R. Aumann and A. Brandenburger, “Epistemic Conditions for Nash Equilibrium,” *Econometrica*, 63, 1995, 1161-1180

A Digression on Quantum Mechanics

Empirical model:

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	f_1	f_2	f_3	f_4
(a’, b)	f_5	f_6	f_7	f_8
(a, b’)	f_9	f_{10}	f_{11}	f_{12}
(a’, b’)	f_{13}	f_{14}	f_{15}	f_{16}

Bell model:

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	1/2	0	0	1/2
(a’, b)	3/8	1/8	1/8	3/8
(a, b’)	3/8	1/8	1/8	3/8
(a’, b’)	1/8	3/8	3/8	1/8

J. Bell, “On the Einstein-Podolsky-Rosen Paradox,” *Physics*, 1, 1964, 195-200

Phase Space

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	f_1	f_2	f_3	f_4
(a', b)	f_5	f_6	f_7	f_8
(a, b')	f_9	f_{10}	f_{11}	f_{12}
(a', b')	f_{13}	f_{14}	f_{15}	f_{16}

	a	a'	b	b'
p_0	0	0	0	0
p_1	0	0	0	1
p_2	0	0	1	0
p_3	0	0	1	1
p_4	0	1	0	0
p_5	0	1	0	1
p_6	0	1	1	0
p_7	0	1	1	1
p_8	1	0	0	0
p_9	1	0	0	1
p_{10}	1	0	1	0
p_{11}	1	0	1	1
p_{12}	1	1	0	0
p_{13}	1	1	0	1
p_{14}	1	1	1	0
p_{15}	1	1	1	1

From Classical to Non-Classical

We cannot assume that the same facts about agreement and disagreement between Bayesian agents hold when they observe non-classical phenomena

A recent physics paper by Frauchiger and Renner (2018) highlights this matter

From an epistemic game theory perspective, their striking claim is that it is possible to have a scenario of “singular disagreement”

> *Alice is certain of an event* E*, and Alice is certain Bob is certain of the complementary event* E^c

Here, Alice is certain of an event F if she assigns probability 1 to F, conditional on her private information

D. Frauchiger and R. Renner, “Quantum Theory Cannot Consistently Describe the Use of Itself,” *Nature Communications*, 9, 2018, 3711

Disagreement in a Non-Classical World

How far can disagreement between agents go in a non-classical world?

We establish three results:

In a non-classical domain, and as in the classical domain, it cannot be common knowledge that two agents assign different probabilities to an event of interest

In a non-classical domain, and unlike the classical domain, it can be common certainty that two agents assign different probabilities to an event of interest

In a non-classical domain, it cannot be common certainty that two agents assign different probabilities to an event of interest, if communication of their common certainty is possible — even if communication does not take place

Summary:

Taken together, the results establish a basic consistency of the non-classical world (like that for the classical world)

General Set-up

There is a finite abstract state space Ω

Alice and Bob have partitions $\mathscr{P}_A$ and $\mathscr{P}_B$ of Ω representing their private information

There is a common (possibly signed) prior probability measure p on Ω

Assume throughout that all members of the partitions $\mathscr{P}_A$ and $\mathscr{P}_B$ receive non-zero probability so that conditioning is well-defined

Singular Disagreement — Classical

Observation: *Suppose that* p *is non-negative and fix an event* E. *Let* G *be the event that Bob assigns probability* 0 *to* E, *i.e.*

$$G = \{\omega' \in \Omega : p(E \,|\, \mathscr{P}_B(\omega')) = 0\}$$

Then there is no state ω *at which Alice assigns probability* 1 *to* $E \cap G$

As a warm-up let’s find singular disagreement in a non-classical setting, using signed probabilities

(See Abramsky and Brandenburger [2011] for a characterization of phase space with signed probabilities)

S. Abramsky and A. Brandenburger, “The Sheaf-Theoretic Structure of Non-Locality and Contextuality,” *New Journal of Physics*, 13, 2011, 113036

Singular Disagreement — Non-Classical

Alice’s (Bob’s) partition is red (blue)

The event of interest is

$$E = \{\omega_1, \omega_3, \omega_4\}$$

The true state is ω_1

At ω_1, Alice assigns (conditional) probability 1 to E

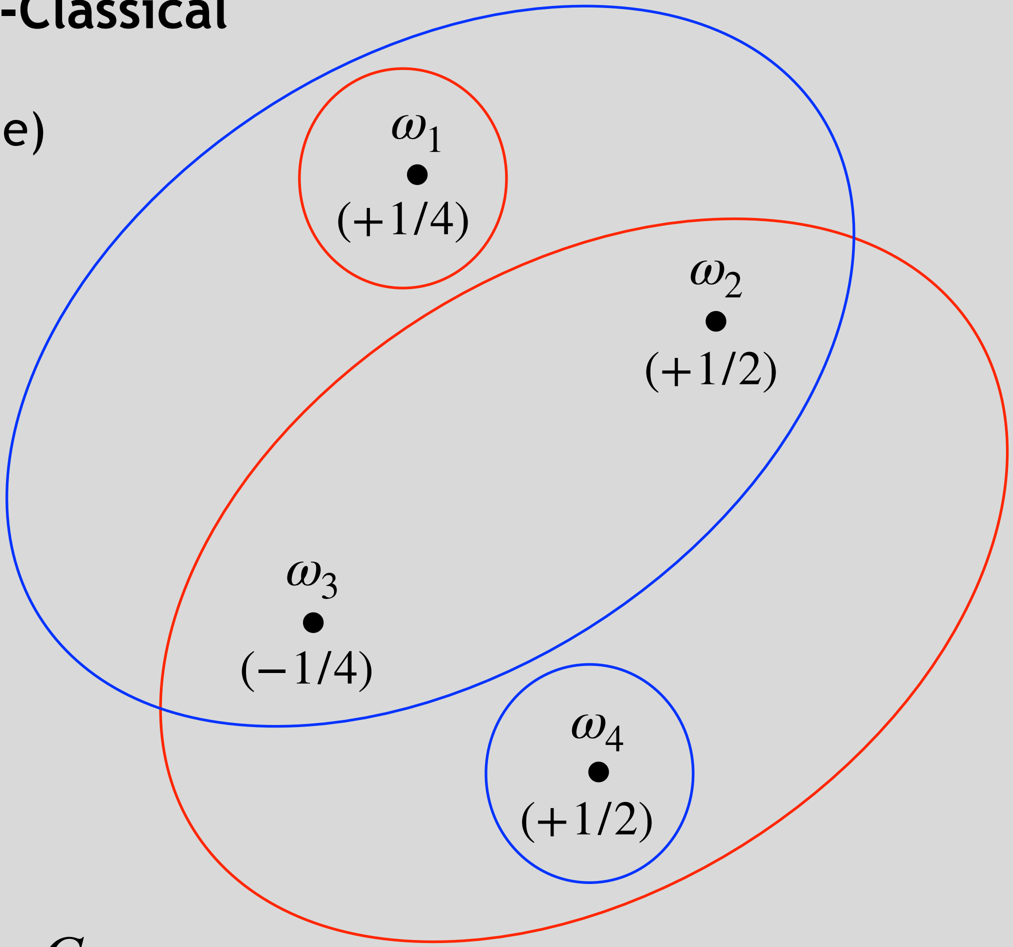
At ω_1, Bob assigns (conditional) probability 0 to E

The event that Bob assigns probability 0 to E is

$$G = \{\omega_1, \omega_2, \omega_3\}$$

At ω_1, Alice assigns probability 1 to G

So, there is singular disagreement!



Note: All partition cells (and events in the join) and E receive strictly positive probability and are therefore observable

Common Certainty

We focus on certainty vs. knowledge

Fix an event E and probabilities q_A and q_B , and let

$$A_0 = \left\{\omega \in \Omega : p(E \mid \mathscr{P}_A(\omega)) = q_A\right\}$$

$$B_0 = \left\{\omega \in \Omega : p(E \mid \mathscr{P}_B(\omega)) = q_B\right\}$$

$$A_{n+1} = A_n \cap \left\{\omega \in \Omega : p(B_n \mid \mathscr{P}_A(\omega)) = 1\right\}$$

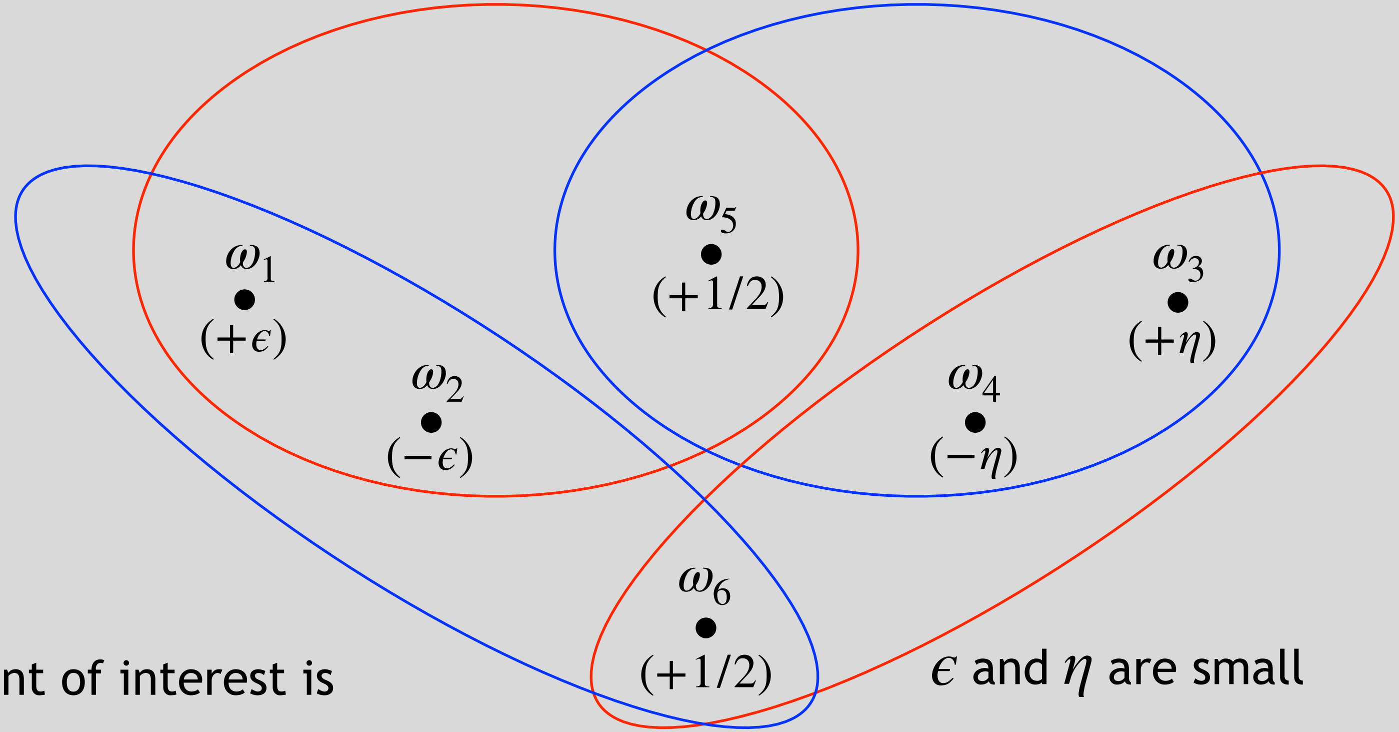
$$B_{n+1} = B_n \cap \left\{\omega \in \Omega : p(A_n \mid \mathscr{P}_B(\omega)) = 1\right\}$$

for all $n \geq 0$

It is *common certainty* at a state ω^* that Alice assigns probability q_A to E and Bob assigns probability q_B to E if

$$\omega^* \in \bigcap_{n=0}^{\infty} A_n \cap \bigcap_{n=0}^{\infty} B_n$$

Common Certainty of Disagreement



ϵ and η are small

The event of interest is

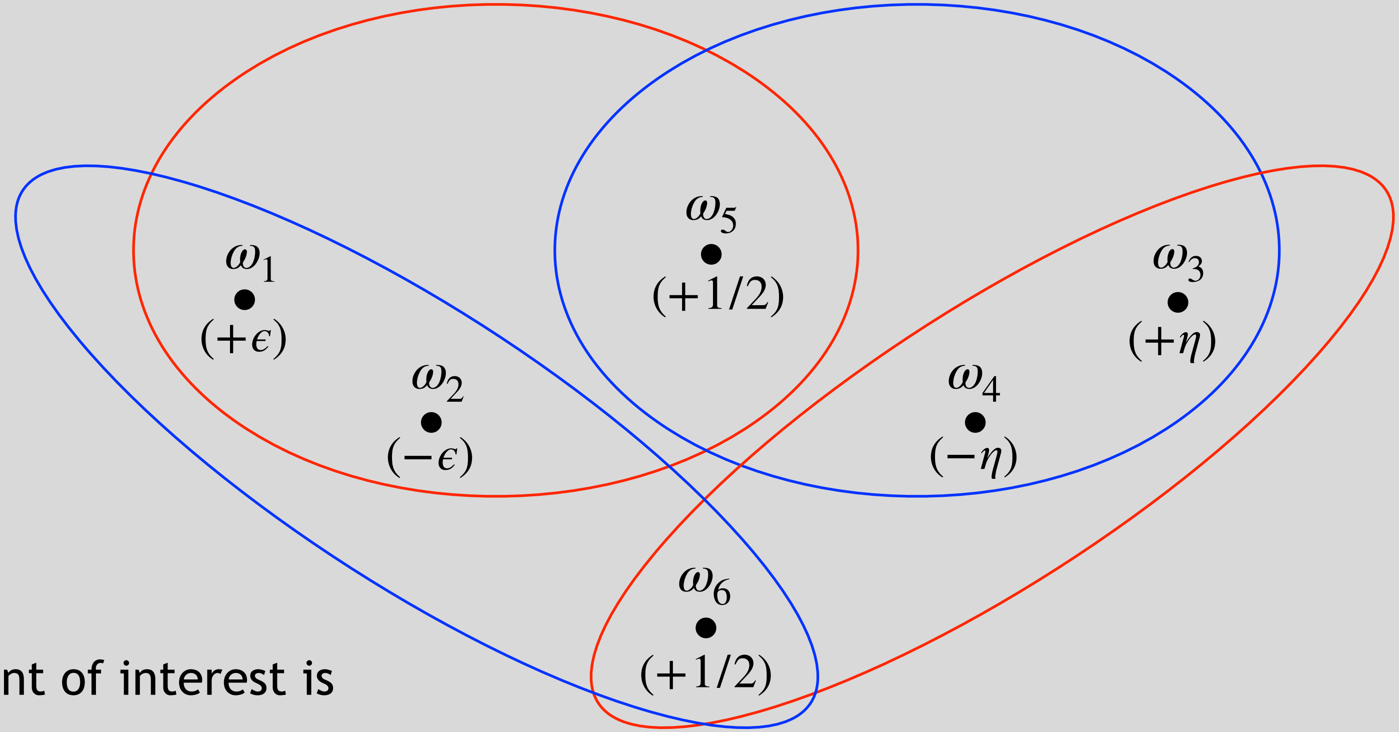
$$E = \{\omega_2, \omega_4, \omega_5, \omega_6\}$$

The true state is ω_5

At ω_5, it is common certainty that Alice assigns probability $1 - 2\epsilon$ to E while Bob assigns probability $1 - 2\eta$ to E

Common certainty of disagreement (just like common knowledge of disagreement) is impossible classically!

Communication



The event of interest is

$$E = \{\omega_2, \omega_4, \omega_5, \omega_6\}$$

The true state is ω_1

Alice communicates her probability to Bob, which tells him she has information $\{\omega_1, \omega_2, \omega_5\}$

Bob's information is then $\{\omega_1, \omega_2\}$, so he forms a (new) probability of $-\epsilon/0$, which is not well-defined!

Communication-Enabled Structures

Define a sequence of partitions for Alice, corresponding to announcements she could make about her probability of E, her certainty of Bob's probability, etc., and likewise for Bob

$$\mathscr{M}_A^{(n)} = \{A_n, A_n^c\}$$
$$\mathscr{M}_B^{(n)} = \{B_n, B_n^c\}$$

For any $\pi, E \subseteq \Omega$, say π is *regular with respect to* E if $p(\pi) \geq 0$ and $0 \leq p(\pi \cap E) \leq p(\pi)$

A structure $(\Omega, p, \mathscr{P}_A, \mathscr{P}_B)$ is *communication-enabled with respect to* E if for each $n \geq 0$, each $\pi \in \mathscr{P}_A \vee \mathscr{M}_B^{(n)}$ and each $\pi \in \mathscr{P}_B \vee \mathscr{M}_A^{(n)}$ is regular with respect to E

Note: This property fails in the previous example

Impossibility of Disagreement Again

Theorem: *Fix a structure that is communication-enabled with respect to* E *and suppose at a state* ω^* *it is common certainty that Alice's probability of* E *is* q_A *and Bob's probability of* E *is* q_B*. Then* $q_A = q_B$

Interestingly, the mere availability of information (here, the information is the common certainty of disagreement) is enough to rule out disagreement — the information need not be observed

There is a variant of the theorem where Alice and Bob are able to communicate with a third agent Charlie, but not with each other

Conclusions

Our results establish a new kind of non-classical strangeness in the form of the possibility of common certainty of disagreement

However, we also prove that common certainty of disagreement under (potential) communication is impossible, even in non-classical settings

Thus, we establish a basic consistency of the non-classical world (like that for the classical world)