Agreement and Disagreement in a Non-Classical World

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The Classical Agreement Theorem

Alice and Bob possess a common prior probability distribution on a state space

They each then receive different private information about the true state

They form their conditional (posterior) probabilities q_A and q_B of an underlying event of interest

Theorem (Aumann, 1976): *If these two values* q_A *and* q_B *are common knowledge between Alice and Bob, they must be equal*

Here, an event E is common knowledge between Alice and Bob if they both know it, both know they both know it, and so on indefinitely

R. Aumann, “Agreeing to Disagree,” *Annals of Statistics*, 4, 1976, 1236-1239

Applications

The agreement theorem is considered a basic requirement in classical epistemics

It has been used to

show that two risk-neutral agents, starting from a common prior, cannot agree to bet with each other

(Sebenius and Geanakoplos, 1983)

prove "no-trade" theorems for efficient markets

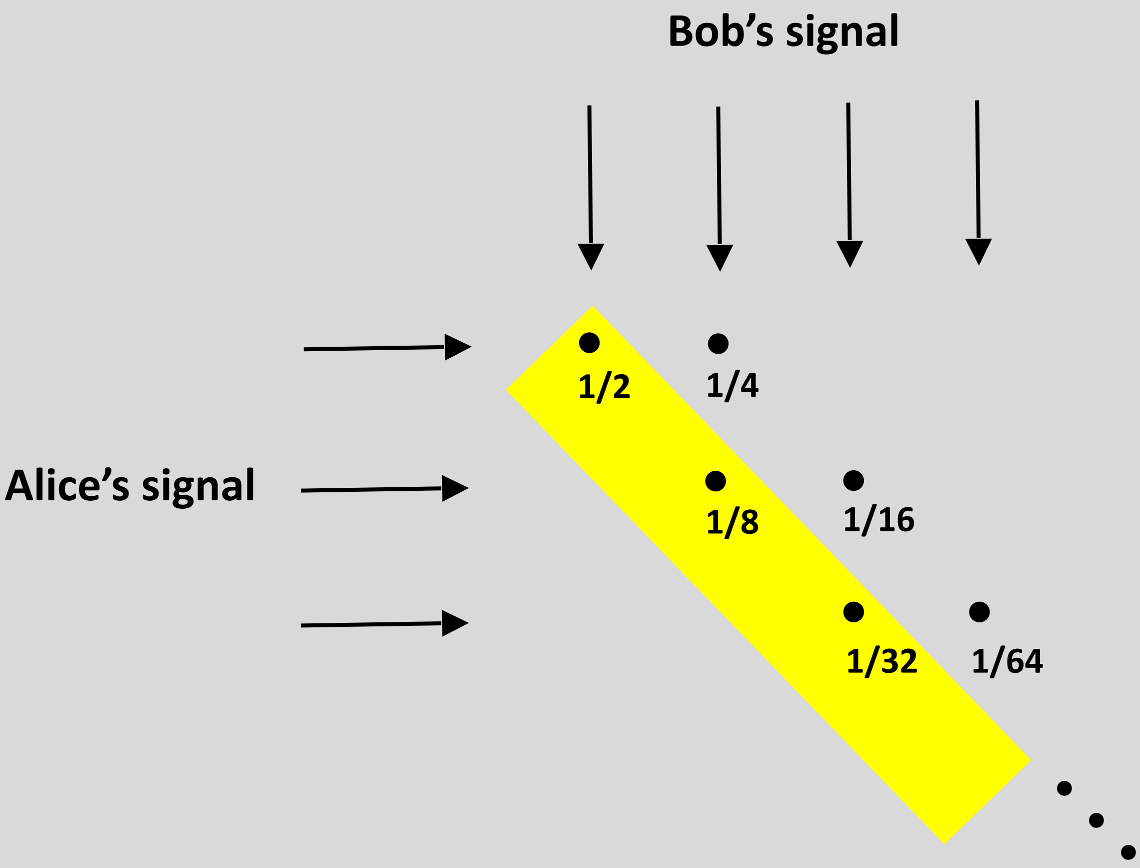
(Milgrom and Stokey, 1982)

establish epistemic conditions for Nash equilibrium

(Aumann and Brandenburger, 1995)

J. Sebenius and J. Geanakoplos. "Don't Bet on It: Contingent Agreements with Asymmetric Information, *Journal of the American Statistical Association*, 78, 1983, 424-426; P. Milgrom and N. Stokey, "Information, Trade, and Common Knowledge," *Journal of Economic Theory*, 26, 1982, 17–27; R. Aumann and A. Brandenburger, "Epistemic Conditions for Nash Equilibrium," *Econometrica*, 63, 1995, 1161–1180

The Role of Common Knowledge: A "Discontinuity" at Infinity



J. Geanakoplos and H. Polemarchakis, "We Can't Disagree Forever," *Journal of Economic Theory*, 28, 1982, 192–200; this variant is due to John Geanakoplos (private communication)

Non-Classical Settings

What is the status of the agreement theorem when classical probability theory does not apply?

The canonical case is the quantum domain, where a fundamental result (Bell's Theorem, 1964) says that no "local hidden-variable" theory can model the results of all quantum experiments

This implies that the classical Bayesian model does not apply

In this talk, we won't do any physics, but physics will motivate our departure from the Bayesian model

Well, a hint of the physics …

J. Bell, "On the Einstein Podolsky Rosen Paradox," *Physics Physique Fizika*, 1964, 195-200

2 x 2 x 2 Boxes

Empirical model:

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	f_1	f_2	f_3	f_4
(a', b)	f_5	f_6	f_7	f_8
(a, b')	f_9	f_{10}	f_{11}	f_{12}
(a', b')	f_{13}	f_{14}	f_{15}	f_{16}

Bell model:

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	1/2	0	0	1/2
(a', b)	3/8	1/8	1/8	3/8
(a, b')	3/8	1/8	1/8	3/8
(a', b')	1/8	3/8	3/8	1/8

Phase Space

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	f_1	f_2	f_3	f_4
(a', b)	f_5	f_6	f_7	f_8
(a, b')	f_9	f_{10}	f_{11}	f_{12}
(a', b')	f_{13}	f_{14}	f_{15}	f_{16}

	a	a'	b	b'
p_0	0	0	0	0
p_1	0	0	0	1
p_2	0	0	1	0
p_3	0	0	1	1
p_4	0	1	0	0
p_5	0	1	0	1
p_6	0	1	1	0
p_7	0	1	1	1
p_8	1	0	0	0
p_9	1	0	0	1
p_{10}	1	0	1	0
p_{11}	1	0	1	1
p_{12}	1	1	0	0
p_{13}	1	1	0	1
p_{14}	1	1	1	0
p_{15}	1	1	1	1

An Impossibility Result

From phase space and the Bell empirical model, we can calculate

$$p_0 + p_1 + p_4 + p_5 = \frac{1}{2},$$

$$p_4 + p_5 + p_{12} + p_{13} = \frac{1}{8},$$

$$p_1 + p_3 + p_5 + p_7 = \frac{1}{8},$$

$$p_0 + p_2 + p_8 + p_{10} = \frac{1}{8}.$$

Adding the second, third, and fourth equations gives

$$p_0 + p_1 + p_2 + p_3 + p_4 + 2p_5 + p_7 + p_8 + p_{10} + p_{12} + p_{13} = \frac{3}{8},$$

which contradicts the first equation.

Signed Probabilities

> "Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative of money."
>
> -- Paul Dirac

Do negative probabilities have uses outside of quantum mechanics?

There is work (e.g., La Mura, 2009; Busemeyer and Bruza, 2012; and Haven, Khrennikov, Ma, and Sozzo, 2018) constructing non-classical decision theories with quantum features to resolve standard paradoxes of choice

Perhaps, there is also route to resolving paradoxes via signed probabilities?

Theorem (Abramsky and Brandenburger, 2011): An empirical model is "no signaling" if and only if there is a phase-space model with a signed probability measure that induces it

P. Dirac, "On the Physical Interpretation of Quantum Mechanics," *Proceedings of the Royal Society of London*, Series A, 180, 1942, 1-40; P. La Mura, "Projective Expected Utility," *Journal of Mathematical Psychology*, 53, 2009, 408-414; J. Busemeyer and P. Bruza, *Quantum Models of Cognition and Decision*, Cambridge University Press, 2012; E. Haven, A. Khrennikov. C. Ma, and S. Sozzo, "Introduction to Quantum Probability Theory and its Economic Applications," *Journal of Mathematical Economics*, 78, 2018, 127-130; S. Abramsky and A. Brandenburger, "The Sheaf-Theoretic Structure of Non-Locality and Contextuality," *New Journal of Physics*, 13, 2011, 113036

General Set-up

There is a finite abstract state space Ω

Alice and Bob have partitions $\mathscr{P}_A$ and $\mathscr{P}_B$ of Ω representing their private information

There is a common – possibly signed – prior probability measure p on Ω

Observability: Assume throughout that all members of the partitions $\mathscr{P}_A$ and $\mathscr{P}_B$ receive probability in the interval $(0,1]$. Assume, too, that all events of interest receive probability in $(0,1]$.

Singular Disagreement — Non-Classical

Alice's (Bob's) partition is red (blue)

The event of interest is

$$E = \{\omega_1, \omega_3, \omega_4\}$$

The true state is ω_1

At ω_1, Alice assigns (conditional) probability 1 to E

At ω_1, Bob assigns (conditional) probability 0 to E

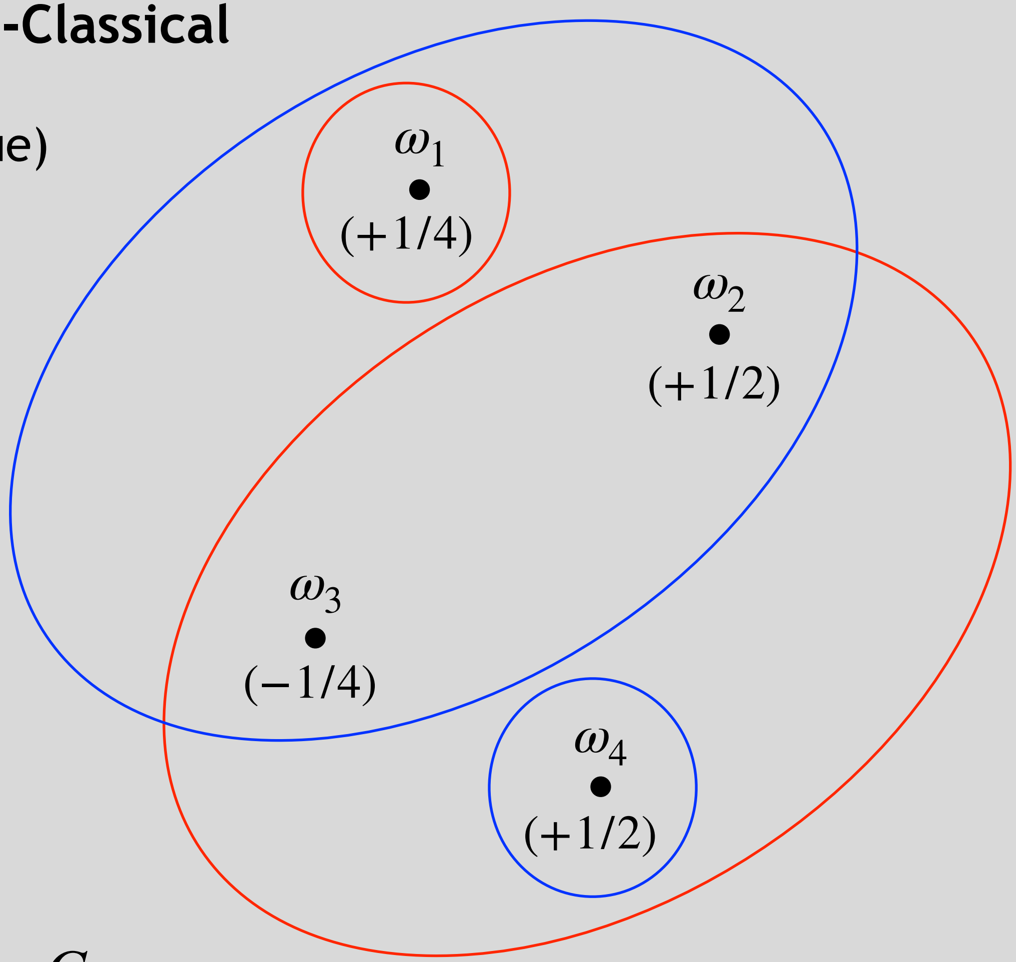
The event that Bob assigns probability 0 to E is

$$G = \{\omega_1, \omega_2, \omega_3\}$$

At ω_1, Alice assigns probability 1 to G

Call this singular disagreement

It is impossible classically!



Note: All partition cells (even events in the join) and E receive strictly positive probability and are therefore observable

From Knowledge to Certainty

Definition: Alice *knows* an event E at state ω if $\mathscr{P}_A(\omega) \subseteq E$

Definition: Alice is *certain of* an event E at a state ω if $p(E \,|\, \mathscr{P}_A(\omega)) = 1$

Fix an event E and probabilities q_A and q_B, and let

$$A_0 = \left\{\omega \in \Omega : p(E \mid \mathscr{P}_A(\omega)) = q_A\right\}$$

$$B_0 = \left\{\omega \in \Omega : p(E \mid \mathscr{P}_B(\omega)) = q_B\right\}$$

$$A_{n+1} = A_n \cap \left\{\omega \in \Omega : p(B_n \mid \mathscr{P}_A(\omega)) = 1\right\}$$

$$B_{n+1} = B_n \cap \left\{\omega \in \Omega : p(A_n \mid \mathscr{P}_B(\omega)) = 1\right\}$$

for all $n \geq 0$

Definition: It is *common certainty* at a state ω^* that Alice assigns probability q_A to E and Bob assigns probability q_B to E if $\omega^* \in \cap_{n=0}^{\infty} A_n \cap \cap_{n=0}^{\infty} B_n$

Relationship Between Knowledge and Certainty

If Alice knows an event E at state ω, then she is certain of E at ω

It is also true that common knowledge of E implies common certainty of E

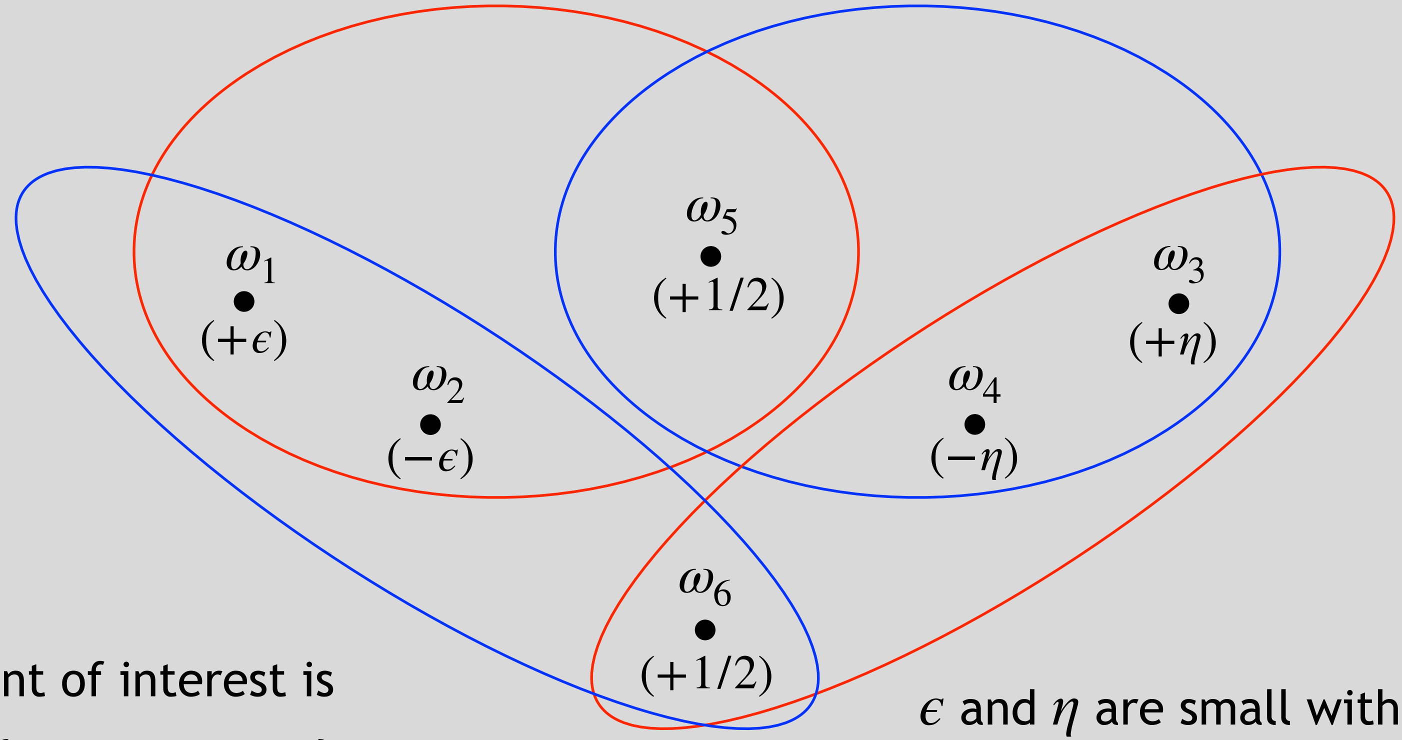
(Proof: If Alice knows Bob knows E, then she knows Bob is certain of E, since knowledge is monotonic. From this, Alice is certain Bob is certain of E. The argument can be continued to all higher levels.)

Arguably, the distinction between these modalities is "small" in the classical domain — at least, in the sense that there is also an agreement theorem for common certainty

Theorem (classical): Fix a (non-negative) common prior and an event E. Suppose at a state ω^* it is common certainty that Alice's probability of E is q_A and Bob's probability of E is q_B. Then $q_A = q_B$.

But what happens in the non-classical world?

Common Certainty of Disagreement



The event of interest is

$$E = \{\omega_2, \omega_4, \omega_5, \omega_6\}$$

ϵ and η are small with $\epsilon \neq \eta$

The true state is ω_5

At ω_5, it is common certainty that Alice assigns probability $1 - 2\epsilon$ to E while Bob assigns probability $1 - 2\eta$ to E

That is, there is common certainty of disagreement!

Realizability of the Counterexample?

Does a scenario of common certainty of disagreement arise in a physical setting?

Physics distinguishes among

i. classical correlations

ii. quantum correlations

iii. superquantum or no-signaling correlations

in 2 x 2 x 2 boxes as earlier (or in larger boxes)

It turns out that common certainty of disagreement — formulated for boxes — is impossible in domain ii. though possible in domain iii.

Can the impossibility of agreeing to disagree thus be elevated to an (epistemic) physical principle?

P. Contreras-Tejada, G. Scarpa, A. Kubicki, A. Brandenburger, and P. La Mura, "Observers of Quantum Systems Cannot Agree to Disagree," *Nature Communications*, 12, 2021, at https://doi.org/10.1038/ s41467-021-27134-6

Realizability of the Counterexample Contd.

Could a scenario of common certainty of disagreement arise in a decision-theoretic setting?

Here, we think of the decision makers as equipped with signed probability measures (as suggested earlier)

This would lead to highly non-classical behavior, such as betting between risk-neutral agents …

Or, should the impossibility of agreeing to disagree be elevated to an (epistemic) decision-theoretic principle?

If yes, what non-classical behavior is then allowed? An open direction …