Reevaluating the Shapley Value: Uniqueness of the α-Procedure

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Inspired by the bargaining procedure of Shapley (1953), Brandenburger and Nalebuff (2023) introduced a novel procedure in which the marginal contribution of the player joining a coalition is split in proportion $\alpha\colon 1-\alpha$ between that player and the members of the coalition being joined. This α-procedure was shown to lead to the Shapley value. Here we demonstrate that our α-procedure is the unique generalized procedure that does so under a priority and a monotonicity axiom.

Keywords: Shapley value, n-person bargaining, generalized bargaining procedures, value division.

In Shapley's (1953) procedure, a player joining an existing coalition keeps all of its marginal contribution. In our procedure, the player joining the existing coalition shares the gains with the members of the coalition in the proportion $\alpha\colon 1-\alpha$. In Brandenburger and Nalebuff (2023), we demonstrated that the α-procedure also leads to the Shapley value. Here we show that our α-procedure is the unique generalized procedure that does so under a priority and a monotonicity axiom.

There are two parts to the α-procedure. The first is the $\alpha\colon 1-\alpha$ split between the joining player and the existing players. The second part is the mechanism for how the $1-\alpha$ share is divided up among members of the existing coalition. An obvious approach would be to divide this share equally. However, such a division would not reflect the differing bargaining positions of the members of the coalition. Even a dummy player, one that adds no value, would get the same amount as all the other members. Our approach divides the $1-\alpha$ share in stages. Each member has a claim equal to their marginal contribution, where the marginal contribution is to the expanded set consisting of the existing members and the joining player. We begin with equal division, but only up to the smallest claim among the members sharing the gains. After that point, the member with the smallest claim no longer shares in the division. We divide the remaining share equally among the remaining members, but only up to the second-smallest

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claim. And so on. Finally, if no member of the existing coalition has a claim to the full $1-\alpha$ share, the residual reverts to the joining player. We call this the α-procedure, defined more formally in Equation (2) below.

Fix a player set N. For each subset $S \subseteq N$, there is a real number $v(S)$ which is the value created by the players in S. We choose the normalization $v(i) = 0$ for all i, and we assume that v is super-additive; that is, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$.

Let the coalition S without player $i \in S$ be denoted by $S\backslash i$. Denote the marginal contribution of a player i in S to coalition S by $m_i(S) = v(S) - v(S\backslash i)$. By super-additivity, $m_i(S) \geq 0$. Let the number of players in coalition S be denoted by $|S|$. We adopt the following labeling convention: for each coalition S, order the players according to their marginal contributions to S, so that $m_1(S) \leq m_2(S) \leq \cdots \leq m_{|S|}(S)$. If two players have equal marginal contributions, then order them arbitrarily. Note that this ordering will generally be different across different coalitions S. To reduce notion, we often refer to $m_i(S)$ by m_i when the meaning is clear.

In our procedure, like the Shapley procedure, players join existing coalitions in a random order. Unlike the Shapley procedure, the gains are split $\alpha\colon 1-\alpha$ between the joiner and the existing coalition members, where $0 \leq \alpha \leq 1$. Instead of summing up across all $|N|!$ possible orderings, we focus on the $|S|$ possible orderings for the last step in how set S can be formed.

In the Shapley procedure, each player $i \in S$ receives $m_i(S)$ when joining $S\backslash i$ and receives zero in the other $|S-1|$ cases where a player $k \in S\backslash i$, joins i in $S\backslash k$. On average, player i receives $m_i(S)/|S|$. In our procedure, each player $i \in S$ receives $\alpha m_i(S)$ when joining $S\backslash i$ and receives a share of $(1-\alpha)m_k(S)$ when player k joins $S\backslash k$.

It remains to specify the division of the amount $(1-\alpha)m_i$ among the $|S|-1$ players in $S\backslash i$. We divide the $(1-\alpha)m_i$ value equally among the players who have a claim to it. To be precise, player i's marginal contribution m_i is divided into i intervals $(0, m_1]$, $(m_1, m_2]$, $\ldots$, $(m_{i-1}, m_i]$. The kth interval $m_k - m_{k-1}$ is then divided equally among the players with $m_j \geq m_k$. There are $|S|-k$ players who have a claim to that interval. Thus, player $j \in S\backslash i$ gets its share on each interval up to the smaller of its maximum claim m_j and the total amount m_i to go around. Formally, player j receives:

$$(1) \qquad (1-\alpha) \sum_{k=1}^{\min(i,j)} \frac{m_k - m_{k-1}}{|S|-k}.$$

In the first term of the sum, we introduced a fictitious player 0 who is not a member of any coalition, and we set $m_0(S) = 0$ for all S. This is done to simplify the notation.

Start with player 1, the player with the lowest marginal contribution. This player gets $(1-\alpha)m_1/(|S|-1|)$ no matter who the outside player is. This exhausts player 1's claim, and the remaining value is split among one fewer players. Player 2 gets the same $(1-\alpha)m_1/(|S|-1)$ as player 1 plus an additional $(1-\alpha)(m_2 - m_1)/(|S|-2)$ when the joiner is anyone other than player 1. Here, the amount $m_2 - m_1$ is divided equally among the $|S|-2$ players whose marginal contributions exceed m_1. This procedure continues for all subsequent players in $S\backslash i$.

When the joining player is $i = |S|$, the procedure works just as before, except that none of the players $j \in S\backslash|S|$ have a claim on the final interval of value $m_{|S|} - m_{|S|-1}$. This amount therefore reverts to player $|S|$. We can see this last step in equation (1), since for any player $j \in S\backslash|S|$, we have $\min(|S|, j) = j < |S|$ and so the interval $m_{|S|} - m_{|S|-1}$ does not appear in the sum for any player j.

The players in $S\backslash i$ have "priority" over the amount $(1-\alpha)m_i$. These players receive this full amount except when $i = |S|$, in which case there is a final interval of value that reverts to i. Putting everything together, our procedure specifies that when player i joins the coalition $S\backslash i$, player j gets a payoff $\pi_{j|i}$ given by:

$$\pi_{j|i}(S) = \begin{cases} \alpha m_j & \text{for } i = j < |S|; \\ \alpha m_{|S|} + (1-\alpha)(m_{|S|} - m_{|S|-1}) & \text{for } i = j = |S|; \\ (1-\alpha) \displaystyle\sum_{k=1}^{\min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} & \text{for } i \neq j. \end{cases} \tag{2}$$

We called this the **α-procedure**. A feature of this procedure is that no player gets more than its marginal contribution to a coalition. In particular, a dummy player gets zero.

Here we want to define a more general weighting system which includes the α-procedure as a special case. Let weights $w_{jk}(S\backslash i) \geq 0$ determine each player j's share of the kth interval of $(1-\alpha)m_i$. We set $w_{jk}(S\backslash i) = 0$ for $k > i$, since there are only i intervals.

Since these are weights, we have

$$\sum_{j \in S} w_{jk}(S\backslash i) = 1. \tag{3}$$

Summing over intervals, we find that each of the players $j \in S\backslash i$ being joined receives

$$(1-\alpha) \sum_{k=1}^{i} w_{jk}(S\backslash i)(m_k - m_{k-1}), \tag{4}$$

and player i receives

$$\alpha m_i + (1-\alpha) \sum_{k=1}^{i} w_{ik}(S\backslash i)(m_k - m_{k-1}). \tag{5}$$

Our first axiom is Priority, which gives the players in $S\backslash i$ precedence in dividing up the $(1-\alpha)m_i$. The very nature of the $\alpha{:}\,1-\alpha$ division is that the players in the coalition being joined should be the ones exclusively sharing the $(1-\alpha)m_i$. The one case in which the players being joined do not receive this full amount is when the largest marginal contribution among players in $S\backslash i$ falls short of m_i. In that case, none of players in $S\backslash i$ have a claim on the full amount of m_i. This arises only when the joining party is $|S|$, the player with the highest marginal contribution. In this exceptional case, the joining party has a unique claim on the final interval, $m_{|S|}(S) - m_{|S|-1}(S)$.

> **Priority Axiom:** The weights $w_{ik}(S\backslash i) = 0$ for all i, k except when $i = k = |S|$.

We also assume that players in $S\backslash i$ with larger marginal contributions get at least as much of each interval as do players with smaller marginal contributions:

> **Monotonicity Axiom:** The weights $w_{jk}(S\backslash i)$ are weakly increasing in $j \in S\backslash i$.

In our α-procedure, the weights, which we write as $w^{\alpha}_{jk}(S\backslash i)$, are given by

$$(6) \qquad w^{\alpha}_{jk}(S\backslash i) = \begin{cases} \dfrac{1}{|S|-k} & \text{for } j \neq i, k \leq \min(i,j); \\ 1 & \text{for } i = j = k = |S|; \\ 0 & \text{otherwise.} \end{cases}$$

These weights satisfy the Priority and Monotonicity axioms.

THEOREM 1: *Under Priority and Monotonicity, the unique weights that lead to the Shapley value are the $w^{\alpha}_{jk}(S\backslash i)$ weights.*

PROOF: Consider a set S with $|S| \geq 3$.[1] We calculate the expected payoff to player $j \in S$ as S goes from size $|S|-1$ to $|S|$. Either j is on the outside and joins $S\backslash j$, or j is on the inside and shares in the gains as the set grows from $S\backslash i$ to S.

When player i joins $S\backslash i$, player j receives

$$(7) \qquad (1-\alpha)\sum_{k=1}^{i} w_{jk}(S\backslash i)(m_k - m_{k-1}).$$

[1]The cases of $|S| = 1$ and $|S| = 2$ do not require any allocation across the players in $S\backslash i$. If $|S| = 1$, there is no one else to share the total value. If $|S| = 2$, there is only one player inside $S\backslash j$ and thus no allocation problem among the players being joined: $w_{11}(S\backslash 2) = w_{21}(S\backslash 1) = 1$. Note also in this case that $m_1(S) = m_2(S)$, so that there is only one relevant interval.

The overall expected gain to player j is thus

$$\frac{1}{|S|}\left[\alpha m_j + \sum_{i\in S}(1-\alpha)\sum_{k=1}^{i} w_{jk}(S\backslash i)(m_k - m_{k-1})\right]. \tag{8}$$

The Shapley value payoff for j is a specific function that depends only on player j's marginal contributions; see Young (1985). Therefore, a necessary and sufficient condition for a generalized weighting system to result in the Shapley value is for all $j \in S$,

$$\alpha m_j + \sum_{i\in S}(1-\alpha)\sum_{k=1}^{i} w_{jk}(S\backslash i)(m_k - m_{k-1}) = m_j. \tag{9}$$

Since the weights are independent of α, we can assume $\alpha < 1$. Thus, equation (9) implies

$$\sum_{i\in S}\sum_{k=1}^{i} w_{jk}(S\backslash i)(m_k - m_{k-1}) = m_j. \tag{10}$$

It remains to show there is a unique set of non-negative weights that satisfy Priority, Monotonicity, and equation (10).

We can rewrite equation (10) as

$$\sum_{k=1}^{|S|} m_k \left[\sum_{i=k}^{|S|} w_{jk}(S\backslash i) - \sum_{i=k+1}^{|S|} w_{jk+1}(S\backslash i)\right] = m_j. \tag{11}$$

Since only m_j appears on the right-hand side, the terms multiplying m_k must be 0 except for $k = j$, when the term must be 1. This implies:

$$\sum_{i=k}^{|S|} w_{jk}(S\backslash i) = \sum_{i=k+1}^{|S|} w_{jk+1}(S\backslash i) \qquad \text{for } k \neq j, \tag{12}$$

and

$$\sum_{i=j}^{|S|} w_{jj}(S\backslash i) = \sum_{i=j+1}^{|S|} w_{jj+1}(S\backslash i) + 1. \tag{13}$$

A recursive argument leads to the simplification of equations (A6) and (A7).

First note that for $k = |S|$,

$$\sum_{i=|S|}^{|S|} w_{j|S|}(S\backslash i) = 0 \qquad \text{for } j < |S| \tag{14}$$

since the right-hand side of equation (12) has no terms in this case. Since the sum in equation (14) is 0 and each term is non-negative, each term must be 0: $w_{j|S|}(S\backslash i) = 0$ for $j < |S|$.

Next consider $k = |S| - 1$. For $j < |S| - 1$,

$$\sum_{i=|S|-1}^{|S|} w_{j|S|-1}(S\backslash i) = \sum_{i=|S|}^{|S|} w_{j|S|}(S\backslash i) = 0, \tag{15}$$

where the first equality follows from equation (12) and the second from equation (14). Furthermore, each term $w_{j|S|-1}(S\backslash i) = 0$, since the sum is 0 and each term is non-negative.

Each time the right-hand side of equation (12) is 0, we use this to show that the left-hand side must be 0 and thus each term must also be 0 for one lower value of k. This continues until $k = j + 1$. The result is Lemma 1.

LEMMA 1: $w_{jk}(S\backslash i) = 0$ *for* $j < k$.

For $k = j$, we have from equation (13) and Lemma 1,

$$\sum_{i=j}^{|S|} w_{jj}(S\backslash i) = \sum_{i=j+1}^{|S|} w_{jj+1}(S\backslash i) + 1 = 0 + 1 = 1. \tag{16}$$

For $k < j$, we return to the recursion argument starting with equations (12) and (13), only now the sum is 1 instead of 0. Starting at $k = j - 1$,

$$\sum_{i=k}^{|S|} w_{jk}(S\backslash i) = \sum_{i=k+1}^{|S|} w_{jk+1}(S\backslash i) = 1, \tag{17}$$

where the right-hand side is 1 by equation (16). Continuing the recursion leads to

$$\sum_{i=k}^{|S|} w_{jk}(S\backslash i) = 1 \qquad \text{for all } k \leq j. \tag{18}$$

For $k = j = |S|$, this implies

$$w_{|S||S|}(S\backslash |S|) = 1. \tag{19}$$

For all other cases with $k \leq j$, we have $w_{jk}(S\backslash j) = 0$ by Priority. Thus, we can rewrite equation (18) as

$$\sum_{i=k, i\neq j}^{|S|} w_{jk}(S\backslash i) = 1 \qquad \text{for } k \leq j < |S| \text{ and } k < j = |S|. \tag{20}$$

In the case of $j = k$, this simplifies to

$$\sum_{i=k+1}^{|S|} w_{kk}(S\backslash i) = 1. \tag{21}$$

Up to this point, we have focused on the constraints that follow from the Shapley value calculation. We can also use the fact that the w's are weights to write

$$\sum_{j\in S} w_{jk}(S\backslash i) = 1 \qquad \text{for } k \leq i. \tag{22}$$

We exclude the cases with $k > i$, because $w_{jk}(S\backslash i) = 0$ by definition.

We previously established (Lemma 1) that $w_{jk}(S\backslash i) = 0$ for $j < k$. By Priority, $w_{jk}(S\backslash j) = 0$ except for $j = k = |S|$. Therefore, equation (22) implies

$$\sum_{j=k, j\neq i}^{|S|} w_{jk}(S\backslash i) = 1 \qquad \text{for } k < i. \tag{23}$$

By Monotonicity, $w_{kk}(S\backslash i) \leq w_{jk}(S\backslash i)$ for $k \leq j$, $j \neq i$, and therefore

$$(|S| - k) w_{kk}(S\backslash i) \leq \sum_{j=k, j\neq i}^{|S|} w_{jk}(S\backslash i) = 1 \qquad \text{for } k < i. \tag{24}$$

This implies

$$w_{kk}(S\backslash i) \leq \frac{1}{|S| - k} \qquad \text{for } k < i. \tag{25}$$

If $w_{kk}(S\backslash i) < 1/(|S| - k)$ for some $k < i$, then

$$\sum_{i=k+1}^{|S|} w_{kk}(S\backslash i) < 1, \tag{26}$$

which violates the Shapley value constraint in equation (21). Therefore, we have

$$w_{kk}(S\backslash i) = \frac{1}{|S| - k} \qquad \text{for } k < i. \tag{27}$$

By Monotonicity, $w_{kk}(S\backslash i) \leq w_{jk}(S\backslash i)$ for $j \geq k$, $j \neq i$. Thus, equation (27) implies:

$$w_{jk}(S\backslash i) \geq w_{kk}(S\backslash i) = \frac{1}{|S| - k} \qquad \text{for } j \neq i,\ k \leq \min(i-1, j). \tag{28}$$

This, in turn, implies

$$w_{jk}(S\backslash i) = \frac{1}{|S| - k} \qquad \text{for } j \neq i,\ k \leq \min(i-1, j), \tag{29}$$

since otherwise the sum in equation (23) would be strictly greater than 1.

At this point, the proof is nearly complete. All that is missing is the case where $k = \min(i, j) = i$, $j \neq i$, which simplifies to $k = i < j$. To solve for this case, we return to equation (20) and substitute in the value $w_{jk}(S\backslash i)$ from equation (29) for all but the first term in the sum. This leads to

$$w_{jk}(S\backslash k) + \frac{|S| - k - 1}{|S| - k} = 1 \qquad \text{for } k < j. \tag{30}$$

Collecting terms, this implies

$$w_{jk}(S\backslash k) = \frac{1}{|S| - k} \qquad \text{for } k < j. \tag{31}$$

Therefore

$$w_{jk}(S\backslash i) = \frac{1}{|S| - k} \qquad \text{for } j \neq i,\ k \leq \min(i, j). \tag{32}$$

Putting this together with equation (19) and Lemma 1, we obtain

$$w_{jk}(S\backslash i) = \begin{cases} \frac{1}{|S| - k} & \text{for } j \neq i,\ k \leq \min(i, j); \\ 1 & \text{for } i = j = k = |S|; \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

as required. ■

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