Reevaluating the Shapley Value:
The NTU Case

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Abstract

This is an extension of "Reevaluating the Shapley Value: A New Justification and Extension" to include the NTU case. All the notation and setup are in our earlier paper.

We show how our α-procedure can be used to define an NTU game. For $\alpha = 1$, this problem has been addressed in Hart and Mas-Colell (1996), who provide a procedure that yields the consistent Shapley value (Maschler and Owen, 1989, 1992). When $0 \leq \alpha < 1$, a new procedure is required, just as in the TU case, since we need a rule to share the $1 - \alpha$ portion of value. We do this by defining the NTU marginal contribution of a player to a set S, which leads to our generalized procedure.

Given an NTU game (N, V), we assume that the feasible sets $V(S)$ satisfy the standard conditions on the characteristic function; see, in particular, conditions (A.1)–(A.3) in Hart and Mas-Colell (1996). Let $\partial V(S)$ denote the boundary of the feasible set for S. For convenience, we perform two normalizations. We set $\partial V(i) = \{0\}$. We also scale the utilities for all players so that the maximum feasible utility level of each player i in $V(N)$ is 1.

We begin with the case where $\partial V(N)$ is a hyperplane (therefore, the unit simplex under our scaling) and then show how to extend our analysis to the general convex case as in Maschler and Owen (1989, 1992). Let Ψ denote the vector of payoffs from the procedure. By our normalization $\Psi(i) = 0$ for one-player games.

Assume inductively that we have a solution for coalitions of size up to $|N| - 1$ (for any characteristic function). Fix a game (N, V). We derive the solution for the set N. The marginal contributions associated with N are

$$d^i(N) := \max\{c^i : (c^i, \Psi(N \backslash i)) \in V(N)\}. \tag{1}$$

This is the maximal possible payoff to i given that the other players obtain their payoffs in the game without i.[1]

With our set of marginal contributions, and following our earlier numbering convention, we index the players in order of increasing marginal contribution. The inductive step in our NTU procedure is obtained by adapting our earlier TU game.

From the set N we randomly select a player to be at risk. Given player i is at risk, we assign the probability $\tau_{j\,i}$ of player j being the proposer as in Brandenburger and Nalebuff (2023), Equation (2), substituting the $d^i(S)$ for the $m_i(S)$, and then dividing by $d^i(S)$.

$$\tau_{j\,i}(S) = \begin{cases} \alpha & \text{for } i = j < S \;; \\ \alpha + \frac{(1-\alpha)}{d_S}\,[d_S - d_{S-1}] & \text{for } i = j = S \;; \\ \frac{(1-\alpha)}{d_i} \sum_{k=1}^{\min(i,j)} \frac{d_k - d_{k-1}}{S - k} & \text{for } i \neq j\,. \end{cases} \tag{2}$$

In that way, the parameter α enters into the procedure.

The procedure (contingent on the random selection of i and j) assigns everyone their value in $\Psi(N \backslash i)$, with the proposer j receiving an additional $d^i(N)$.[2] Because $\partial V(N)$ is a hyperplane (normalized to the unit simplex), it is always efficient and feasible to assign $d^i(N)$ to the player making the proposal. The payoffs $\Psi(N)$ are the expected values where each player has an equal chance of being at risk,

$$\Psi(N) = \frac{1}{N} \sum_{i=1}^{N} [\Psi(N \backslash i) + \sum_{j=1}^{N} \tau_{ji} d^i(N \backslash i) e^j]. \tag{3}$$

[1] This definition is different from that in Hart and Mas-Colell (1996) in two ways. First, the marginal contributions defined in Equation (B1) are independent of the order of player arrival. Second, outside of a hyperplane game, the solution to the subgame $\Psi(N \backslash i)$ need not be the average marginal contribution of each player in that game. In Hart and Mas-Colell, the marginal contributions are defined inductively based on a specific ordering of player arrivals. For a hyperplane game—one in which $V(S)$, for $S \subseteq N$, is a half space—our $d^i(N)$ equal the average value of the Hart and Mas-Colell marginal contributions across all $(N-1)!$ possible orderings that build to N in which player i is the last to join.

[2] As before, we extend $\Psi(N \backslash i)$ so that player i—who is not part of $N \backslash i$—receives 0 in $\Psi(N \backslash i)$.

and where e^j is the jth unit vector. Again, because $\partial V(N)$ is a hyperplane, this expected value is both efficient and feasible.

Moreover, the same argument as in Equation (3) in the proof of Theorem 1 of Brandenburger and Nalebuff (2023) shows

$$\frac{1}{N}\sum_{i=1}^{N}\tau_{ji}d^i(N\backslash i) = d^j(N\backslash i). \tag{4}$$

Thus

$$\Psi(N) = d(N) + \frac{1}{N}\sum_{i=1}^{N}\Psi(N\backslash i), \tag{4}$$

where $d(N)$ is the vector of the $d^j(N)$. We can see that this procedural solution is the NTU analog to the Shapley recursion relationship.

However, this is only the solution for the case where $\partial V(N)$ is a hyperplane. To find the procedural solution for general $V(N)$, we look for a fixed point as in Maschler and Owen (1992). Start with a point p on the unit simplex. Consider the ray from the origin through p. This ray will intersect $\partial V(N)$ at some point q. Let hyperplane $H(q)$ be tangent to $\partial V(N)$ at q. Normalize $H(q)$ so that it is the unit simplex, and apply the same scaling to $V(N)$. Consider the game when the scaled $V(N)$ is extended to $H(q)$. Here, the boundary is a hyperplane, so we can apply the solution for $\Psi(N)$ from Equation (B4). This is a continuous mapping from the unit simplex to itself—from p to q to $\Psi(N)$—and thus has a fixed point. The fixed point is a tangency point and thus on the boundary of (the scaled) $V(N)$. At the fixed point, the solution for $\Psi(N)$ is then defined as the consistent solution to the feasible set $V(N)$. The intuition for selecting the fixed point is similar to the axiom of Independence of Irrelevant Alternatives: $\Psi(N)$ is a solution for a larger set that includes $V(N)$ and it remains feasible in the smaller set $V(N)$, so it should be the solution in the smaller set.

Observe that the inductive step has two parts. We start with N players and randomly break the set into $N-1$ insiders and one "at-risk" player. We apply the procedure to a game with $N-1$ players, and divide up the at-risk player's contribution to obtain the solution to a game with N players. This first step is done when the boundary for $V(N)$ is a hyperplane. We then use the solution to all such games to find a fixed point for general $V(N)$. This is similar to the way the Nash bargaining solution is constructed.

We offer some remarks on our NTU procedure. First, if the game is TU, the procedure leads to the same result as our α-procedure defined in Equations (1)–(2). Next, for two-person games, our NTU procedure leads to the Nash (1950) bargaining solution for all values of α. When the boundary of the bargaining set is a line, the NTU procedure selects the midpoint: $\Psi(N) = 1/2[(\alpha,1-\alpha) + (1-\alpha,\alpha)] = (1/2,1/2)$. As in the Nash bargaining solution, the NTU procedure for convex sets selects the boundary point which is the midpoint of the tangent line at that boundary point.

For $\alpha = 1$, our procedure leads to the same consistent solution(s) as in Hart and Mas-Colell.[3] Any consistent solution is based on the solution to a hyperplane game and our procedures align in hyperplane games when $\alpha = 1$.

References

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[3] Hart and Mas-Colell allow for a penalty in the case of disagreement that we set to 0. The solutions coincide when the penalty is 0.