

Agreement between observers: a physical principle?

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We prove the results in the main text. We also provide the statements and some definitions in the interest of readability.

Theorem 1. *Fix a probability space $(\Omega, \mathcal{E}, \mathbb{P})$ where E_A and E_B are perfectly correlated events. If it is common certainty at a state $\omega^* \in \Omega$ that Alice assigns probability q_A to E_B and Bob assigns probability q_B to E_A , then $q_A = q_B$.*

Proof. Since Ω is finite, there is a finite $N \in \mathbb{N}$ such that, for all $n \geq N$, $A_{n+1} = A_n$ and $B_{n+1} = B_n$. From the definition of A_{N+1} , we have that

$$\mathbb{P}(B_N | \mathcal{P}_A(\omega)) = 1 \quad \forall \omega \in A_N. \quad (1)$$

Now, A_N is a union of partition elements of \mathcal{P}_A , i.e., $A_N = \bigcup_{i \in I} \pi_i$ where each $\pi_i \in \mathcal{P}_A$ and I is a finite index set. From Equation (1), we have

$$\mathbb{P}(B_N | \pi_i) = 1 \quad \forall i \in I. \quad (2)$$

Since $\mathbb{P}(B_N | A_N)$ is a convex combination of $\mathbb{P}(B_N | \pi_i)$ for $i \in I$, we must have

$$\mathbb{P}(B_N | A_N) = 1. \quad (3)$$

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Now, since $A_N \subseteq A_0$, then $\mathbb{P}(E_B|\pi_i) = q_A$ for all $i \in I$ too. Using a convex combination argument once more, this entails that

$$\mathbb{P}(E_B|A_N) = q_A. \quad (4)$$

Equations (3) and (4) together imply that

$$\mathbb{P}(E_B|A_N \cap B_N) = q_A. \quad (5)$$

But events E_A and E_B are perfectly correlated, so that

$$\mathbb{P}(E_A \cap E_B|A_N \cap B_N) = q_A, \quad (6)$$

as well.

Running the parallel argument with A and B interchanged, we obtain

$$\mathbb{P}(E_A \cap E_B|A_N \cap B_N) = q_B, \quad (7)$$

which implies that $q_A = q_B$. \square

Proposition A1. *Given any no-signaling box $\{p(ab|xy)\}_{(a,b,x,y) \in \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y}}$, there is a (non-unique) corresponding ontological model whose probabilities assigned to the states of the world are not necessarily non-negative.*

This result was already derived in [1] from sheaf-theoretic concepts, however we provide a much more direct proof that is more suitable for the purposes of this work.

Proof. Let $\{p(ab|xy)\}_{a,b,x,y}$ be a no-signaling box. We construct its associated ontological model $\{(\Omega, \mathcal{F}, \mathbb{P}), \{A_x^a, B_y^b\}_{a,b,x,y}\}$. We provide the proof for $a, b, x, y \in \{0, 1\}$ for ease of notation, but the generalization to more inputs and outputs is immediate.

To construct the ontological model, we postulate the existence of a set of states

$$\omega_{a_0 a_1 b_0 b_1} \quad (8)$$

with quasi-probabilities

$$\mathbb{P}_{a_0 a_1 b_0 b_1} \equiv \mathbb{P}(\omega_{a_0 a_1 b_0 b_1}). \quad (9)$$

Each state corresponds to an *instruction set* [2], i.e., the state where Alice outputs a_0 on input $x = 0$ and a_1 on input $x = 1$, and Bob outputs b_0 on input $y = 0$ and b_1 on input $y = 1$. Then, each $\mathbb{P}_{a_0 a_1 b_0 b_1}$ is the quasi-probability of the corresponding instruction set. Of course, if the given box is post-classical, not all of these quasi-probabilities will be non-negative. In fact, in principle it need not even be guaranteed that one can find a quasi-probability distribution over these

states. But we will use the probability distribution of the inputs and outputs of the given no-signaling box to derive a linear system of equations over the quasi-probabilities, and show that it does have a solution.

There are 16 states in total, as there are two possible outputs for each of the 4 inputs ($|\mathcal{A}|^{|\mathcal{X}|} \cdot |\mathcal{B}|^{|\mathcal{Y}|}$ in general). Then, each partition corresponds to a set of states as follows:

$$\begin{aligned} \mathbf{A}_x^a &= \{\omega_{a_0 a_1 b_0 b_1} : a_x = a\} \\ \mathbf{B}_y^b &= \{\omega_{a_0 a_1 b_0 b_1} : b_y = b\} . \end{aligned} \tag{10}$$

We associate the probabilities $p(ab|xy)$ of the no-signaling box to the probabilities $\mathbf{P}(\mathbf{A}_x^a \cap \mathbf{B}_y^b)$ of each intersection of partitions, for each input pair (x, y) and output pair (a, b) . This gives rise to a set of equations for the probabilities $\mathbf{P}_{a_0 a_1 b_0 b_1}$. Indeed, the probability of each intersection is given by

$$\mathbf{P}(\mathbf{A}_x^a \cap \mathbf{B}_y^b) = \sum_{a_{\bar{x}}, b_{\bar{y}}} \mathbf{P}_{a_x a_{\bar{x}} b_y b_{\bar{y}}} \tag{11}$$

where we denote the output corresponding to the input that is not x as $a_{\bar{x}}$, and similarly for $b_{\bar{y}}$, and so we have, for each a, b, x, y ,

$$\sum_{a_{\bar{x}}, b_{\bar{y}}} \mathbf{P}_{a_x a_{\bar{x}} b_y b_{\bar{y}}} = p(ab|xy) . \tag{12}$$

Since there are 16 values of $p(ab|xy)$ in the 2-input 2-output no-signaling box, we arrive at 16 equations ($|\mathcal{A}| \times |\mathcal{B}| \times |\mathcal{X}| \times |\mathcal{Y}|$ in general). Of course, there are some linear dependencies between the equations, but we will show that the system still has a solution.

The system of equations can be expressed as

$$MP = C \tag{13}$$

where M is the matrix of coefficients, P is the vector of probabilities $\mathbf{P}_{a_0 a_1 b_0 b_1}$ and C is the vector of independent terms $p(ab|xy)$. The system has a solution (which is not necessarily unique) if and only if

$$\text{rank}(M) = \text{rank}(M|C) . \tag{14}$$

Since the rank of a matrix is the number of linearly independent rows, it is trivially true that

$$\text{rank}(M) \leq \text{rank}(M|C) , \tag{15}$$

as including the independent terms can only remove some relations of linear dependence, not add more. Equivalently, the number of relations of linear dependence

of $M|C$ is always smaller than or equal to the number of relations of linear dependence of M . Therefore, to show that their ranks are equal, it is sufficient to show that every relation of linear dependence that we find in M still holds in $M|C$. That is, for every relation of linear dependence between the probabilities $P_{a_0a_1b_0b_1}$ that is contained in M , it is sufficient to show that the relation still holds when the sums of probabilities are matched to the elements $p(ab|xy)$ of the no-signaling box in order to show that the system of equations has a solution.

Observe that M contains only zeros and ones, as the equations (12) are just sums of probabilities. Moreover, each column of M corresponds to the probability of a state $\omega_{a_0a_1b_0b_1}$, while each row corresponds to an equation with independent term $p(ab|xy)$. Because the equations (12) correspond to intersections of partitions of the set of $\omega_{a_0a_1b_0b_1}$, we can observe that each row of M has a 1 in the column corresponding to the states $\omega_{a_0a_1b_0b_1}$ contained in the corresponding partition, and a 0 elsewhere. Put another way, in order to construct M one must first partition the set of $\omega_{a_0a_1b_0b_1}$ in four different ways, corresponding to

$$\begin{aligned} & \{A_0^a\}_a, \{A_1^a\}_a, \\ & \{B_0^b\}_b, \{B_1^b\}_b \end{aligned} \tag{16}$$

for Alice and Bob respectively. This gives partitions of the columns of M . Then, the 16 possible ways of intersecting partitions of Alice's with partitions of Bob's give the 16 equations with independent term $p(ab|xy)$. But notice now that the partition structure imposes a certain relation of linear dependence between the rows of M . Indeed, for each b, y , we have

$$\left\{ \bigcup_a (A_0^a \cap B_y^b) \right\} = \left\{ \bigcup_a (A_1^a \cap B_y^b) \right\}, \tag{17}$$

as

$$\bigcup_a (A_0^a \cap B_y^b) = \left(\bigcup_a A_0^a \right) \cap B_y^b = \Omega \cap B_y^b = \left(\bigcup_a A_1^a \right) \cap B_y^b = \bigcup_a (A_1^a \cap B_y^b), \tag{18}$$

and, similarly, for each a, x we have

$$\left\{ \bigcup_b (A_x^a \cap B_0^b) \right\} = \left\{ \bigcup_b (A_x^a \cap B_1^b) \right\}. \tag{19}$$

Using the correspondence of these partitions with the partitions of the columns of M gives 8 relations of linear dependence between its rows. Now, noticing that a union of columns of M corresponds to a sum of probabilities $P_{a_0a_1b_0b_1}$, we find

that these relations correspond exactly to the no-signaling conditions, as

$$\mathbb{P} \left(\bigcup_a (A_x^a \cap B_y^b) \right) = \sum_a \mathbb{P} (A_x^a \cap B_y^b) \quad (20)$$

so

$$\sum_a \mathbb{P} (A_0^a \cap B_y^b) = \sum_a \mathbb{P} (A_1^a \cap B_y^b) \quad (21)$$

and similarly for Bob. Of course, by definition of no-signaling box, these relations hold for the independent terms $p(ab|xy)$ as well, since

$$\begin{aligned} \sum_a \mathbb{P} (A_x^a \cap B_y^b) &= \sum_a p(ab|xy), \\ \sum_b \mathbb{P} (A_x^a \cap B_y^b) &= \sum_b p(ab|xy) \end{aligned} \quad (22)$$

by construction of the linear system (see equations (11) and (12)). Therefore, every relation of linear dependence between the rows of M holds also between the rows of $M|C$, as required.

Notice also that the implication goes both ways: the linear system has a solution *only if* the set of probabilities $p(ab|xy)$ is no-signaling. The coefficient matrix M incorporates the no-signaling conditions by construction of the states $\omega_{a_0 a_1 b_0 b_1}$ with probabilities $\mathbb{P}_{a_0 a_1 b_0 b_1}$. Therefore, if these conditions do not hold for the independent terms $p(ab|xy)$, then the rank of $M|C$ must be larger than that of M , as $M|C$ contains more linearly independent rows than M . \square

We now state and prove the classical agreement theorem in the no-signaling language, i.e. for local boxes. We restrict to boxes of two inputs and two outputs since, by Theorem 5, any larger box exhibiting disagreement can be reduced to a 2-input 2-output box that also exhibits disagreement, while preserving its locality properties.

Recall that a distribution $\{p(ab|xy)\}_{(a,b,x,y) \in \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y}}$ is local if, for each a, b, x, y ,

$$p(ab|xy) = \sum_{\lambda} p_{\lambda} p_A(a|x\lambda) p_B(b|y\lambda), \quad (23)$$

for some distributions $\{p_{\lambda}\}_{\lambda \in \Lambda}$, $\{p_A(a|x\lambda)\}_{(a,x,\lambda) \in \mathcal{A} \times \mathcal{X} \times \Lambda}$, $\{p_B(b|y\lambda)\}_{(b,y,\lambda) \in \mathcal{B} \times \mathcal{Y} \times \Lambda}$ where Λ is an index set.

Corollary 2. *Suppose Alice and Bob share a local no-signaling box with underlying probability distribution p . Let $q_A, q_B \in [0, 1]$, and let*

$$\begin{aligned} p(b = 1|a = 0, x = 0, y = 1) &= q_A, \\ p(a = 1|b = 0, x = 1, y = 0) &= q_B. \end{aligned} \quad (24)$$

If q_A and q_B are common certainty between the observers, then $q_A = q_B$.

Proof. By definition of q_A, q_B , and using the fact that the shared distribution is local and hence satisfies equation (23), we have

$$\begin{aligned} q_A \sum_{\lambda} p_{\lambda} p_A(0|0\lambda) &= \sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_B(1|1\lambda) \\ q_B \sum_{\lambda} p_{\lambda} p_B(0|0\lambda) &= \sum_{\lambda} p_{\lambda} p_A(1|1\lambda) p_B(0|0\lambda). \end{aligned} \quad (25)$$

From Theorem 3 we have that, if $1 \in \alpha_n$ or $1 \in \beta_n$ for all $n \in \mathbb{N}$ then there is no common certainty of disagreement for any no-signaling distribution, and these encompass local distributions. Hence there only remains to prove the claim for $1 \notin \alpha_n$ and $1 \notin \beta_n$, for some $n \in \mathbb{N}$. This implies that

$$\begin{aligned} p(b = 0|a = 0, x = 0, y = 0) &= 1 \\ p(a = 0|b = 0, x = 0, y = 0) &= 1 \end{aligned} \quad (26)$$

and hence

$$\begin{aligned} \sum_{\lambda} p_{\lambda} p_A(0|0\lambda) &= \sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_B(0|0\lambda) \\ \sum_{\lambda} p_{\lambda} p_B(0|0\lambda) &= \sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_B(0|0\lambda), \end{aligned} \quad (27)$$

which implies, on the one hand, that

$$\sum_{\lambda} p_{\lambda} p_A(0|0\lambda) = \sum_{\lambda} p_{\lambda} p_B(0|0\lambda) \quad (28)$$

and, on the other, that

$$\sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_B(1|0\lambda) = \sum_{\lambda} p_{\lambda} p_A(1|0\lambda) p_B(0|0\lambda) = 0, \quad (29)$$

that is,

$$p_A(0|0\lambda) p_B(1|0\lambda) = p_A(1|0\lambda) p_B(0|0\lambda) = 0 \quad (30)$$

for all λ . Therefore, there remains to prove only that

$$\sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_B(1|1\lambda) = \sum_{\lambda} p_{\lambda} p_A(1|1\lambda) p_B(0|0\lambda). \quad (31)$$

Because the outputs for inputs $x = 1, y = 1$ are perfectly correlated, we have

$$p_A(0|1\lambda) p_B(1|1\lambda) = p_A(1|1\lambda) p_B(0|1\lambda) = 0 \quad (32)$$

for all λ and, since $p_A(0|1\lambda) + p_A(1|1\lambda) = 1$ and similarly for p_B , this implies

$$p_A(1|1\lambda) = p_B(1|1\lambda). \quad (33)$$

Then we can prove (31) by simple manipulations of the probability distributions of each party:

$$\begin{aligned}
\sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_B(1|1\lambda) &= \sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_A(1|1\lambda) (p_B(0|0\lambda) + p_B(1|0\lambda)) \\
&= \sum_{\lambda} p_{\lambda} p_A(0|0\lambda) p_A(1|1\lambda) p_B(0|0\lambda) \\
&= \sum_{\lambda} p_{\lambda} (p_A(0|0\lambda) + p_A(1|0\lambda)) p_A(1|1\lambda) p_B(0|0\lambda) \\
&= \sum_{\lambda} p_{\lambda} (p_A(0|1\lambda) + p_A(1|1\lambda)) p_A(1|1\lambda) p_B(0|0\lambda) \\
&= \sum_{\lambda} p_{\lambda} p_A(1|1\lambda) p_A(1|1\lambda) p_B(0|0\lambda) \\
&= \sum_{\lambda} p_{\lambda} p_A(1|1\lambda) p_B(0|0\lambda)
\end{aligned} \tag{34}$$

where we have used the fact that $\sum_{b \in \mathcal{B}} p_B(b|y\lambda) = 1$ for all y, λ in the first equality, (??) in the second and third, $\sum_{a \in \mathcal{A}} p_A(a|x\lambda) = 1$ for all x, λ in the fourth, (32) again in the fifth, and $p_A(1|1\lambda)^2 = p_A(1|1\lambda)$ for all λ (since $p_A(a|x\lambda)$ is either 1 or 0 for every a, x, λ) in the last. \square

In the proof of Theorem 3, we will make use of the following Lemma:

Lemma A2. *Consider a no-signaling box of 2 inputs and 2 outputs. Then, $\alpha_0 = \{0, 1\}$ if and only if $q_A = p(b = 1|y = 1)$. Analogously, $\beta_0 = \{0, 1\}$ if and only if $q_B = p(a = 1|x = 1)$.*

Proof. By hypothesis,

$$\begin{aligned}
q_A &= p(b = 1|a = 0, x = 0, y = 1) = \frac{p(01|01)}{p(a = 0|x = 0)} \\
&= p(b = 1|a = 1, x = 0, y = 1) = \frac{p(11|01)}{p(a = 1|x = 0)}.
\end{aligned}$$

But now, we can write

$$p(b = 1|y = 1) = p(01|01) + p(11|01) = p(a = 0|x = 0)q_A + p(a = 1|x = 0)q_A = q_A. \blacksquare$$

The reverse implication is trivial. The analogous statement can be proved by interchanging the roles of Alice and Bob. \square

Theorem 3. *A 2-input 2-output no-signaling box gives rise to common certainty of disagreement if and only if it takes the form of Table 1.*

$xy \backslash ab$	00	01	10	11
00	r	0	0	$1 - r$
01	$r - s$	s	$-r + t + s$	$1 - t - s$
10	$t - u$	u	$r - t + u$	$1 - r - u$
11	t	0	0	$1 - t$

Table 1: Parametrization of no-signaling boxes with common certainty of disagreement. Here, $r, s, t, u \in [0, 1]$ are such that all the entries of the box are non-negative, $r > 0$, and $s - u \neq r - t$.

Proof. We first prove that common certainty of disagreement imposes the claimed structure for the no-signaling box. Therefore, we assume common certainty of disagreement, i.e.,

$$(0, 0, 0, 0) \in A_n \cap B_n \quad \forall n \in \mathbb{N}. \quad (35)$$

In particular, we also assume that Alice and Bob input $x = y = 0$ and obtain $a = b = 0$. This implies

$$p(00|00) > 0. \quad (36)$$

We split the proof into three cases based on the contents of the sets A_n, B_n :

Case 1. $1 \notin \alpha_n, 1 \notin \beta_n$ for some n .¹ From common certainty of disagreement (equation (35)), we have that

$$p(B_n|a = 0, x = 0, y = 0) = 1, \quad p(A_n|b = 0, x = 0, y = 0) = 1,$$

which, together with $1 \notin \alpha_n, 1 \notin \beta_n$, translates into:

$$p(01|00) = 0, \quad p(10|00) = 0.$$

We also assumed that the observers in fact obtained outputs $a = 0, b = 0$ on inputs $x = 0, y = 0$, so we must have $p(00|00) > 0$. The rest of the table is determined by no-signaling constraints in terms of parameters r, s, t and u . Given the box in the statement of the theorem, $q_A \neq q_B$ if and only if $s - u \neq r - t$, which concludes the proof of this case.

Case 2. $\alpha_n = \{0, 1\}$, for all $n \in \mathbb{N}$ while $1 \notin \beta_m$ for some m . We show that this case implies $q_A = q_B$, so it contradicts common certainty of disagreement. Indeed, the definition of α_{m+1} enforces the conditions:

$$p(b = 0|a = 0, x = 0, y = 0) = 1 = p(b = 0|a = 1, x = 0, y = 0).$$

¹This need not happen at the same stage, i.e., possibly $1 \notin \alpha_m$, for some $m < n$. However in this case, since the sets are nonempty by assumption, we have $\alpha_n = \alpha_m$.

This implies

$$p(b = 1|a = 0, x = 0, y = 0) = \frac{p(01|00)}{p(a = 0|x = 0)} = 0 \quad \Rightarrow \quad p(01|00) = 0,$$

$$p(b = 1|a = 1, x = 0, y = 0) = \frac{p(11|00)}{p(a = 1|x = 0)} = 0 \quad \Rightarrow \quad p(11|00) = 0.$$

Adding no-signaling conditions to these last equations, we also obtain

$$0 = p(b = 1|y = 0) = p(01|10) + p(11|10), \quad (37)$$

and so

$$p(01|10) = 0 = p(11|10) \quad (38)$$

and

$$p(b = 0|y = 0) = 1. \quad (39)$$

This allows us to identify q_B with $p(a = 1|x = 1)$, since

$$\begin{aligned} q_B &= p(a = 1|b = 0, x = 1, y = 0) \\ &= \frac{p(10|10)}{p(b = 0|y = 0)} \\ &= p(10|10) \\ &= p(a = 1|x = 1) - p(11|10) \\ &= p(a = 1|x = 1), \end{aligned}$$

where the third and last equalities follow from equations (39) and (38) respectively. Now, taking into account Lemma A2 and perfect correlations, we have

$$q_A = p(b = 1|y = 1) = p(a = 1|x = 1),$$

which shows that $q_A = q_B$, as mentioned above.

Case 3. $\alpha_n = \{0, 1\}$, $\beta_n = \{0, 1\}$ for all $n \in \mathbb{N}$. We now show that this case also implies $q_A = q_B$, contradicting common certainty of disagreement. Using Lemma A2 we have

$$q_B = p(a = 1|x = 1) \quad \text{as well as} \quad q_A = p(b = 1|y = 1).$$

Now, perfect correlations impose that $p(a = 1|x = 1) = p(b = 1|y = 1)$, that is, $q_A = q_B$.

Next, we prove the converse implication of the theorem. We show that any no-signaling box of the above form must exhibit common certainty of disagreement. Since $s - u \neq r - t$, we have that Alice and Bob assign different probabilities to output $a, b = 1$ on input $x, y = 1$:

$$\begin{aligned} q_A &= p(b = 1|a = 0, x = 0, y = 1) = s/r, \\ q_B &= p(a = 1|b = 0, x = 1, y = 0) = (r - t + u)/r. \end{aligned} \tag{40}$$

In the case that $1 \notin \alpha_0$, $1 \notin \beta_0$, we also have that $\alpha_1 = \alpha_0$ and $\beta_1 = \beta_0$, and common certainty of disagreement follows, because $(0, 0, 0, 0)$ is in $A_n \cap B_n$ for all n .

If the parameters are such that

$$\frac{1 - t - s}{1 - r} = \frac{s}{r}, \tag{41}$$

but

$$\frac{1 - r - u}{1 - r} \neq \frac{r - t + u}{r}, \tag{42}$$

then

$$p(b = 1|a = 1, x = 0, y = 1) = q_A, \tag{43}$$

as well, but

$$p(a = 1|b = 1, x = 1, y = 0) \neq q_B, \tag{44}$$

and so $1 \in \alpha_0$, $1 \notin \beta_0$. Since we have

$$p(b = 0|a = 0, x = 0, y = 0) = 1, \tag{45}$$

we find $(0, 0, 0, 0) \in A_1$,² and hence all A_n still contain $(0, 0, 0, 0)$, yielding common certainty of disagreement.

Symmetric reasoning covers the case $1 \notin \alpha_0$, $1 \in \beta_0$, and only the case where $\alpha_0 = \{0, 1\}$, $\beta_0 = \{0, 1\}$ remains. This happens when

$$\begin{aligned} p(b = 1|a = 1, x = 0, y = 1) &= p(b = 1|a = 0, x = 0, y = 1), \\ p(a = 1|b = 0, x = 1, y = 0) &= p(a = 1|b = 1, x = 1, y = 0) \end{aligned} \tag{46}$$

which, in terms of the parameters, is equivalent to

$$\frac{1 - t - s}{1 - r} = \frac{s}{r}, \tag{47}$$

$$\frac{1 - r - u}{1 - r} = \frac{r - t + u}{r}. \tag{48}$$

²Note $(1, 0, 0, 0) \notin A_1$, though this does not affect the present proof.

However, these two conditions are satisfied simultaneously only when $s - u = r - t$, as we now show. From Equation (47) we get

$$s = r(1 - t),$$

while from Equation (48) we obtain

$$u = t(1 - r).$$

This means that if Equations (47) and (48) are both satisfied, then

$$s - u = r(1 - t) - t(1 - r) = r - t,$$

which contradicts the statement of the Theorem. \square

Theorem 5. *No quantum box can give rise to common certainty of disagreement.*

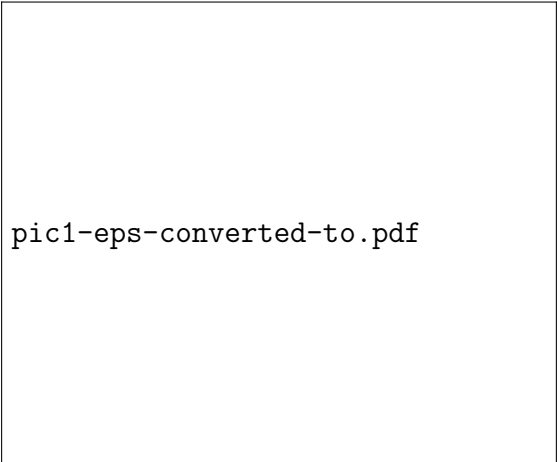
Proof. We show that any no-signaling box with common certainty of disagreement induces a 2-input 2-output no-signaling box with the same property. Thus, if there existed a quantum system that could generate the bigger box, it could also generate the smaller box. Then, Theorem 3 implies that no quantum box can give rise to common certainty of disagreement.

We define a mapping from a distribution $\{p(ab|xy)\}_{a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{X}, y \in \mathcal{Y}}$ to an ‘effective’ distribution $\{\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\}_{\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y} \in \{0,1\}}$ such that the following conditions hold:

1. if $\{p(ab|xy)\}$ is quantum, then so is $\{\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\}$,
2. if $\{p(ab|xy)\}$ satisfies common certainty of disagreement, then so does $\{\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\}$.

First, notice that the number of inputs can be reduced to 2 without loss of generality, as common certainty of disagreement is always defined to be *at* an event (wlog, $(0, 0, 0, 0)$) *about* another event (wlog, $(1, 1, 1, 1)$). One can associate the inputs $x = 0, y = 0$ with $\tilde{x} = 0, \tilde{y} = 0$, respectively, and $x = 1, y = 1$ with $\tilde{x} = 1, \tilde{y} = 1$ respectively, and ignore all other possible inputs in \mathcal{X}, \mathcal{Y} . The outputs, instead, must be grouped according to whether or not they belong in the sets A_n, B_n (for input 0) and whether or not they correspond to the event obtaining, i.e. whether or not they are equal to 1 (for input 1).

Since p satisfies common certainty of disagreement, we know that $(0, 0, 0, 0) \in A_n \cap B_n$. Moreover, by the definitions of the sets α_n, β_n (and since we only consider



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Figure 1: A diagrammatic representation of the construction of \tilde{p} .

finite sets $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$) there exists an $N \in \mathbb{N}$ such that $\alpha_n = \alpha_N$ and $\beta_n = \beta_N$ for all $n \geq N$. Take such N , and define the following indicator functions:

$$\begin{aligned} \chi_{0|0}^\alpha(a) &= \begin{cases} 0 & a \notin \alpha_N \\ 1 & a \in \alpha_N \end{cases} \\ \chi_{0|0}^\beta(b) &= \begin{cases} 0 & b \notin \beta_N \\ 1 & b \in \beta_N \end{cases} \\ \chi_{0|1}^\alpha(c) = \chi_{0|1}^\beta(c) &= \begin{cases} 0 & c = 1 \\ 1 & c \neq 1 \end{cases} \end{aligned} \quad (49)$$

(where c stands for output a, b for Alice and Bob, respectively), with

$$\begin{aligned} \chi_{1|x}^\alpha(a) &= 1 - \chi_{0|x}^\alpha(a) \\ \chi_{1|y}^\beta(b) &= 1 - \chi_{0|y}^\beta(b) \end{aligned} \quad (50)$$

for each a, b, x, y . Then, the mapping from p to \tilde{p} is defined as follows:

$$\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}) = \sum_{a,b} \delta_{x,\tilde{x}} \delta_{y,\tilde{y}} \chi_{\tilde{a}|x}^\alpha(a) \chi_{\tilde{b}|y}^\beta(b) p(ab|xy) \quad (51)$$

where

$$\delta_{s,t} = \begin{cases} 0 & s \neq t \\ 1 & s = t. \end{cases} \quad (52)$$

We note that the distribution \tilde{p} is merely a local post-processing of p , and hence it is quantum if p is, and local if p is. Indeed, the function χ that defines \tilde{p} only

relates the inputs and outputs of each observer individually, and, if p is local, so is \tilde{p} , as

$$\begin{aligned}
\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}) &= \sum_{a,b} \delta_{x,\tilde{x}}\delta_{y,\tilde{y}}\chi_{\tilde{a}|x}^\alpha(a)\chi_{\tilde{b}|y}^\beta(b) \sum_{\lambda} p_{\lambda}p_A(a|x\lambda)p_B(b|y\lambda) \\
&= \sum_{\lambda} p_{\lambda} \left(\sum_a \delta_{x,\tilde{x}}\chi_{\tilde{a}|x}^\alpha(a)p_A(a|x\lambda) \right) \left(\sum_b \delta_{y,\tilde{y}}\chi_{\tilde{b}|y}^\beta(b)p_B(b|y\lambda) \right) \\
&= \sum_{\lambda} p_{\lambda}\tilde{p}_A(\tilde{a}|\tilde{x}\lambda)\tilde{p}_B(\tilde{b}|\tilde{y}\lambda)
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
\tilde{p}_A(\tilde{a}|\tilde{x}\lambda) &= \sum_a \delta_{x,\tilde{x}}\chi_{\tilde{a}|x}^\alpha(a)p_A(a|x\lambda) \\
\tilde{p}_B(\tilde{b}|\tilde{y}\lambda) &= \sum_b \delta_{y,\tilde{y}}\chi_{\tilde{b}|y}^\beta(b)p_B(b|y\lambda)
\end{aligned} \tag{54}$$

for each $\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}, x, y, \lambda$. The fact that \tilde{p} is quantum can be proven by a similar transformation of the measurement operators that gave rise to p .

Also, it is possible to prove explicitly that normalization and no-signaling are preserved by the transformation. Assuming that p is normalized and no-signaling, normalization follows straightforwardly from the definition, since for each input, each output in p gets mapped to a unique output in \tilde{p} , and all of the outputs in p get mapped to some output in \tilde{p} (i.e. the map from p to \tilde{p} is a surjective function). Because the map is defined differently for each pair of inputs and outputs, the no-signaling conditions need to be checked for each line. However, the computations all follow the same pattern, and we perform only one as an example:

$$\begin{aligned}
\sum_{\tilde{a}} \tilde{p}(\tilde{a}0|00) &= \sum_{\substack{a \in \alpha_N \\ b \in \beta_N}} p(ab|00) + \sum_{\substack{a \notin \alpha_N \\ b \in \beta_N}} p(ab|00) \\
&= \sum_{\substack{a \in \mathcal{A} \\ b \in \beta_N}} p(ab|00) \\
&= \sum_{\substack{a \in \mathcal{A} \\ b \in \beta_N}} p(ab|10) \\
&= \sum_{\substack{a \neq 1 \\ b \in \beta_N}} p(ab|10) + \sum_{b \in \beta_N} p(1b|10) \\
&= \sum_{\tilde{a}} \tilde{p}(\tilde{a}0|10)
\end{aligned} \tag{55}$$

where we have used the no-signaling property of p in the third line, and the rest follows from the definition of the map (51).

To check condition 2, let N be as in the definition of the map (51) and let $a \in \alpha_N$. Then, by definition of the set α_{N+1} , we have

$$p(\beta_N|a, x = 0, y = 0) = 1 \quad (56)$$

and, therefore,

$$\frac{\sum_{b \in \beta_N} p(ab|00)}{\sum_{b \in \mathcal{B}} p(ab|00)} = 1, \quad (57)$$

which entails

$$\sum_{b \notin \beta_N} p(ab|00) = 0. \quad (58)$$

Summing over $a \in \alpha_N$, we get

$$\sum_{\substack{a \in \alpha_N \\ b \notin \beta_N}} p(ab|00) = \tilde{p}(01|00) = 0. \quad (59)$$

Similarly, we find $\tilde{p}(10|00) = 0$. Since p satisfies common certainty of disagreement, its outputs on input $x = 1, y = 1$ must be perfectly correlated. That is, $p(ab|11) = 0$ if $a \neq b$. Hence,

$$\tilde{p}(01|11) = \sum_{a \neq 1} p(a1|11) = 0 \quad (60)$$

and similarly for $\tilde{p}(10|11)$. So far, the no-signaling box corresponding to \tilde{p} has two zeros in the first row and another two in the last. Using normalization and no-signaling conditions to fill in the rest of the table, we find it is of the form of the no-signaling box in Theorem 3. There remains to check for disagreement, i.e. that if

$$q_A = p(b = 1|a = 0, x = 0, y = 1) \neq p(a = 1|b = 0, x = 1, y = 0) = q_B \quad (61)$$

then

$$\tilde{p}(\tilde{b} = 1|\tilde{a}, \tilde{x} = 0, \tilde{y} = 1) \neq \tilde{p}(\tilde{a} = 1|\tilde{b}, \tilde{x} = 1, \tilde{y} = 0). \quad (62)$$

Since $\alpha_N \subseteq \alpha_0$ and $\beta_N \subseteq \beta_0$, $p(b = 1|a^*, x = 0, y = 1) \neq p(a = 1|b^*, x = 1, y = 0)$ holds in particular for all $a^* \in \alpha_N, b^* \in \beta_N$. This means that, for $a^* \in \alpha_N, b^* \in \beta_N$,

$$\frac{p(a^*1|01)}{\sum_{b \in \mathcal{B}} p(a^*b|01)} \neq \frac{p(1b^*|10)}{\sum_{a \in \mathcal{A}} p(ab^*|10)} \quad (63)$$

and so

$$p(a^*1|01) \sum_{a \in \mathcal{A}} p(ab^*|10) \neq p(1b^*|10) \sum_{b \in \mathcal{B}} p(a^*b|01). \quad (64)$$

Then, we can sum over α_N and β_N on both sides to find

$$\sum_{a^* \in \alpha_N} p(a^*1|01) \sum_{\substack{a \in \mathcal{A} \\ b^* \in \beta_N}} p(ab^*|10) \neq \sum_{b^* \in \beta_N} p(1b^*|10) \sum_{\substack{a^* \in \alpha_N \\ b \in \mathcal{B}}} p(a^*b|01). \quad (65)$$

But in terms of \tilde{p} , this corresponds to

$$\tilde{p}(01|01) \sum_{\tilde{a} \in \{0,1\}} \tilde{p}(\tilde{a}0|10) \neq \tilde{p}(10|10) \sum_{\tilde{b} \in \{0,1\}} \tilde{p}(0\tilde{b}|01) \quad (66)$$

which implies

$$\tilde{p}(\tilde{b} = 1 | \tilde{a} = 0, \tilde{x} = 0, \tilde{y} = 1) \neq \tilde{p}(\tilde{a} = 1 | \tilde{b} = 0, \tilde{x} = 1, \tilde{y} = 0) \quad (67)$$

and hence the disagreement occurs for the \tilde{p} distribution as well, which proves the result.

Notice that the sets $\tilde{\alpha}_0, \tilde{\beta}_0$ in the distribution \tilde{p} (defined analogously to α_0, β_0 in the distribution p) will correspond to outputs $\tilde{a}, \tilde{b} = 0$, respectively. This is to be expected, as the map $p \rightarrow \tilde{p}$ gives rise to a no-signaling box of the form of the one in Theorem 3, where the sets $\tilde{\alpha}_0, \tilde{\beta}_0$ contain a single element each. (In effect, this means we are ignoring the outputs $a^* \in \alpha_0 \setminus \alpha_N$ and $b^* \in \beta_0 \setminus \beta_N$, but those outputs lead to disagreement but not to common certainty of it, so they can be safely discarded.)

□

Theorem 6. *There is no local 2-input 2-output box that gives rise to singular disagreement.*

Proof. Assume Alice and Bob input $x = y = 0$ and obtain $a = b = 0$. This implies

$$p(00|00) > 0. \quad (68)$$

Alice assigns

$$p(b = 1 | a = 0, x = 0, y = 1) = 1, \quad (69)$$

and Bob assigns

$$p(a = 1 | b = 0, x = 1, y = 0) = 0. \quad (70)$$

Further, the outputs for input $(x, y) = (1, 1)$ are perfectly correlated, so, in particular,

$$p(01|11) = 0. \quad (71)$$

. Equations (69) and (70) imply, respectively,

$$p(00|01) = 0 \text{ and } p(10|10) = 0. \quad (72)$$

However, equations (68), (71) and (72) make up a form of Hardy's paradox [3], which is known not to hold for local distributions. □

Theorem 9. *No quantum box can give rise to singular disagreement.*

Proof. Like in Theorem 5, we show that any no-signaling box with singular disagreement induces a 2-input 2-output no-signaling box with the same property, and rely on Theorem 8 to deduce that no quantum system can give rise to singular disagreement.

Analogously to Theorem 5, to prove the Theorem for singular disagreement we define a mapping from a distribution $\{p(ab|xy)\}_{a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{X}, y \in \mathcal{Y}}$ to an ‘effective’ distribution $\{\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\}_{\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y} \in \{0,1\}}$ such that the following conditions hold:

1. if $\{p(ab|xy)\}$ is quantum, then so is $\{\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\}$,
2. if $\{p(ab|xy)\}$ satisfies singular disagreement, then so does $\{\tilde{p}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\}$.

Again, the number of inputs can be reduced to 2 without loss of generality. To group the outputs, we notice that the sets A_0, B_0 also play a role in singular disagreement, as they group the outputs of each party which lead them to assign their respective probabilities to the event. Then, we group the outputs according to whether or not they belong in the sets α_0, β_0 (for input 0) and whether or not they correspond to the event obtaining, i.e. whether or not they are equal to 1 (for input 1). We obtain the same mapping (51) as before, substituting α_N for α_0 and β_N for β_0 . With this replacement, condition 1 follows by the same proof as before. To check condition 2, we know that, for all $a^* \in \alpha_0$,

$$p(b = 1|a^*, x = 0, y = 1) = 1 \quad (73)$$

and so

$$p(a^*1|01) = \sum_{b \in \mathcal{B}} p(a^*b|01). \quad (74)$$

Summing over $a^* \in \alpha_0$ and rewriting the expression in terms of \tilde{p} , we find

$$\tilde{p}(01|01) = \sum_{\tilde{b} \in \{0,1\}} \tilde{p}(0\tilde{b}|01) \quad (75)$$

which implies

$$\tilde{p}(\tilde{b} = 1|\tilde{a} = 0, \tilde{x} = 0, \tilde{y} = 1) = 1. \quad (76)$$

Similarly, for all $b^* \in \beta_0$ we have

$$p(a = 1|b^*, x = 1, y = 0) = 0, \quad (77)$$

hence

$$p(1b^*|10) = 0 \quad (78)$$

and so, by adding over $b^* \in \beta_0$ and mapping to \tilde{p} , we find

$$\tilde{p}(10|10) = 0 \tag{79}$$

as required. \square

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